STRONG SUMMABILITY OF FOURIER SERIES AND THE SERIES CONJUGATE TO IT

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1.1. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$ and let the Fourier series of $f(t)$ be

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$= \sum_{n=1}^{\infty} A_n(t). \tag{1}$$

Then the conjugate series of $f(t)$ is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt)$$

$$= \sum_{n=1}^{\infty} B_n(t). \tag{2}$$

We write

$$\phi(t) = \frac{\{f(x+t) + f(x-t) - 2f(x)\}}{2},$$

$$\psi(t) = \frac{\{f(x+t) - f(x-t)\}}{2};$$

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$$\theta(t) = \frac{2}{\pi} \int_t^\infty \frac{\psi(u)}{u} du,$$

and suppose throughout that $t > 0$. Evidently the Fourier series of $\Phi(t)$ and $\psi(t)$ are respectively

$$\tfrac{1}{2} a_0 - f(x) + \sum_{n=1}^\infty A_n(x) \cos nt,$$

and

$$\sum_{n=1}^\infty B_n(x) \sin nt.$$

We also write

$$\Phi_a(t) = \frac{1}{\Gamma(a)} \int_0^t (t-u)^{a-1} \Phi(u) du, \; a > 0,$$

$$\phi_0(t) = \phi(t),$$

$$\phi_a(t) = \Gamma(a+1) t^{-a} \Phi_a(t),$$

for $a > 0$. We employ $\Psi_a(t)$ and $\psi_a(t)$ with similar meaning.

1.2. Strong summability of order unity was defined as early as in 1913 by Hardy and Littlewood[9] in context with Cesàro summation of Fourier series. The definition was extended to positive orders in general by Winn[23] in 1933. But the results obtained so far concerning strong summability of Fourier series or the series conjugate to it are mostly confined to the case of unit order, that is summability [C, l; k]. The object of this paper is to study strong summability of Fourier series (1) and the conjugate series (2) for general positive orders. It has also been recently shown that strong Cesàro summability [summability (C, a; k)] is equivalent to strong Rieszian summability of type n [summability (R, n, a, k), $k > 1$].[6, 18] Because of this fact we work with summability [R] throughout instead of summability [C].

1.3. *Definition* 1.[6, 23]—The series Σc_n or the sequence $\{s_n{}^s\}$ is said to be strongly summable (C, a) or summable [C, a], where $a > 0$, to sum s, if

$$\sum_{s=0}^n | s_s{}^{a-1} - s | = o(n),$$

as $n \to \infty$, $s_s{}'$, $r > -1$, denoting Cesàro's partial mean of order r. If instead,

$$\sum_{s=1}^n | s_s{}^{a-1} - s |^s = o(n),$$

the series is said to be summable [C, α] with index p, or simply summable [C, α; p]. Putting O for o we get the corresponding definitions for boundedness. Also

$$\sum_{v=0}^{n} | s_v^{\alpha-1} |^p = o \{n \chi(n)\},$$

as $n \to \infty$, is expressed by

$$\Sigma c_n = o \{\chi(n)\} [C, \alpha; p].$$

Definition 2 [18].—The series Σc_n is said to be strongly summable (R, λ_n, α), or summable [R, λ_n, α], $\alpha > o$, to sum s, if

$$\int^{\omega} | C_\lambda^{\alpha-1}(x) - s | \, dx = o(\omega),*$$

as $\omega \to \infty$, where $C_\lambda^r(x)$ denotes Riesz's mean of order $r > -1$, type λ_n, λ_n being a positive steadily increasing function of n, tending to infinity with n. If

$$\int^{\omega} | C_\lambda^{\alpha-1}(x) - s |^p \, dx = o(\omega),$$

the series is said to be summable [R, λ_n, α, p]. Again, putting O for o in the above we get the corresponding definitions for boundedness. Further,

$$\Sigma c_n = o \{\chi(n)\} [R, \lambda_n, \alpha, p]$$

is supposed to denote

$$\int^{\omega} | C_\lambda^{\alpha-1}(x) |^p \, dx = o \{\omega \chi(\omega)\},$$

as $\omega \to \infty$.

1.4. The result of Hardy and Littlewood gives:

Theorem A [9].—*If* f(t) *is of integrable square in the neighbourhood of* x *and*

$$\int_0^t \{\phi(u)\}^2 \, du = o(t),$$

as t $\to$ o, *then*

$$\sum_{v=0}^{n} | s_v - s |^2 = o(n),$$

as n $\to \infty$, *that is to say* (1) *is summable* [C, 1; 2] *at the point* t = x *to sum* s= f(x).

Generalisations of this theorem were given by Carleman[7] and Sutton.[20] Again in 1927, Hardy and Littlewood[11] established the following theorem, which as they term it, 'has more the air of finality'.

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* The lower limit may be any finite positive number. In such cases, here and elsewhere, we omit the lower limit.

Theorem B.[11]—*If* f (t) *belongs to the class* L^r, r > 1 *in the neighbourhood of x,*

$$\Phi_1 (t) = o (t)$$

and

$$\Phi_1^r (t) = \int_0^t | \phi (u) |^r \, du = O (t)$$

as t $\to$ o, *then the series* (1) *is summable* [C, 1; k] *at the point* t $= x$ *to sum* f (x).

In case $r = 1$, they showed later that even if O is replaced by o, *i.e.*, $\Phi_1^1 (t) = o (t)$, as $t \to o$, then

$$\sum A_n (t) = o \left\{ (\log n)^{k/2} \right\} \quad [C, 1; k], \tag{i}$$

at the point $t = x$, at any rate when $k < 2$, and that this is the best possible estimate [12, Theorems 1 and 3]. Later Wang established:

Theorem C.[22]†—*If, for some* $\alpha > \frac{1}{2}$,

$$\int_0^t | \phi (u) | \, du = o \left\{ \frac{t}{\left(\log \frac{1}{t} \right)^\alpha} \right\},$$

then the Fourier series (1) *of* f (t) *is summable* [C, 1; 2] *to sum* f (x), *at* t $= x$.

Theorem 1 of the present paper is an extension to summability [C, $\alpha + 1$; k] of Theorem B. Theorem 2, similarly, extends Theorem C. Theorem 3 is concerned with asymptotic estimate of a nature similar to (i) of the above, applicable to positive order summability under more general conditions. Theorems 4, 5 and 6 give the analogues of Theorems 1, 2 and 3 for the conjugate series.

· 1·5. A number of theorems is known connecting summability (C, β) with the behaviour of the Cesàro means $\Phi_\alpha (t)$ of the function $\Phi (t)$. One of these gives that if $\phi_\alpha (t) = o (1)$, as $t \to o$, then the Fourier series (1) is summable (C, β), $\beta > \alpha$ [1]. But when we consider strong Cesàro summability, Theorem 1 indicates that under this very condition, (1) is summable [C, $\alpha + 1$; k]. And this throws more light than the above for summability (C), because it is known that [18, Theorem 7] summability [C, $\alpha + 1$; k] for all k (finite) includes summability (C, δ), $\delta > \alpha$.

The first attempt, in the direction of the converse problem of determining the properties of $\phi_\alpha (t)$ from the summability [C] of the Fourier series, is due to Bosanquet who shows that if the series (1) is summable [C, 1] at $t = x$, then $\phi_\delta (t) \to o$, as $t \to o$, for $\delta > 1$ [2 Theorem 1]. He also proved the corresponding result for the conjugate series [2, Theorem 2]. Later using Riesz's arithmetic mean, he gave another theorem which is to the effect that if (1) is summable [R, n, α),‡

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† The correspooding result for the conjugate series has been given by Singh (17).

‡ In view of the Definition 2, above.

or simply summable [C, α], $\alpha > 1$, at $t = x$, then $\phi_\delta(t) = o(1)$, as $t \to o$, for $\delta > \alpha$ [4, Theorem 6]. We extend this and show in Theorem 7 that it will remain true even for $\alpha > o$. Theorem 8 is the analogue of Theorem 7 for the conjugate series.

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2.1. We establish the following theorems:—

Theorem 1§.—If, for $r > 1$,

$$\phi_\alpha^r(t) = \int_0^t |\phi_\alpha(u)|^r \, du = O(t),$$

and

$$\int_0^t \phi_\alpha(u) \, du = o(t),$$

as $t \to o$, then the series (1) is summable [R, n, $\alpha + 1$, k] *at the point* t = x *to sum* f(x).

Theorem 2.—*If, for some* $\beta > \frac{1}{2}$,

$$\int_0^t |\phi_\alpha(u)| \, du = o\left\{\frac{t}{\left(\log \frac{1}{t}\right)^\beta}\right\},$$

as $t \to o$, *then the series* (1) *is summable* [R, n, $\alpha + 1$, 2] *to sum* f(x) *at* t = x.

*Theorem 3.**—*If, for* $\beta > -\frac{1}{2}$,

$$\int_0^t |\phi_\alpha(u)| \, du = o\left\{\frac{t}{\left(\log \frac{1}{t}\right)^\beta}\right\},$$

as $t \to o$, *then*

$$\Sigma A_n(t) = o\{(\log n)^{2\beta+1}\} \ [R, n, \alpha + 1, 2],$$

and when $\beta = -\frac{1}{2}$,

$$\Sigma A_n(t) = o\{\log \log n\} \ [R, n, \alpha + 1, 2].$$

Theorem 4.—*If, for* $r > 1$,

$$\Psi_\alpha^r(t) = \int_0^t |\psi_\alpha(u)|^r \, du = O(t),$$

§ While preparing the manuscript, I have come to know that this theorem was also proved by Takahashi, *vide Jap. Jour. Math.*, 1934, **11**, 213–21. But the proofs given by us are quite different.

* For $\alpha = o$, $\beta = o$, the theorem was given by Hardy and Littlewood [12, Theorem 1], and for $\alpha = o$, $\beta > o$, *see* Cheng (8; Lemma, p. 408).

and

$$\int_0^t \psi_\alpha(u) = o(t),$$

as t → o, *then the conjugate series* (2) *is summable* [R, n, α + 1, k] *to the sum* s *at the point* t = x, *provided* θ (t) → s (C).†

Theorem 5.—*If, for some* β > ½,

$$\int_0^t |\psi_\alpha(u)|\, du = o\left\{\frac{t}{\left(\log\frac{1}{t}\right)^\beta}\right\},$$

as t → o, *then the series* (2) *is summable* [R, n, α + 1, 2] *to the sum* s, *at the point* t = x, *provided* θ (t) → s (C).

Theorem 6.—*If, for* β > − ½,

$$\int_0^t |\psi_\alpha(u)|\, du = o\left\{\frac{t}{\left(\log\frac{1}{t}\right)^\beta}\right\},$$

as t → o, *then the sequence*

$$\{n\, B_n(t)\} = o\{(\log n)^{2\beta+1}\}\ [R, n, \alpha + 2, 2],\ddagger$$

and when β = − ½,

$$\{n\, B_n(t)\} = o(\log\log n)\ [R, n, \alpha + 2, 2].\S$$

at t = x.

Theorem 7.—*If* α > o, *and* (1) *is summable* [R, n, α] *to sum* f (x), *at* t = x, *then* $\phi_\delta(t)$ → o, *as* t → o, *for every* δ > α.

Theorem 8.—*If* α > o, *and* (2) *is summable* [R, n, α] *to sum* s, *at* t = x, *then* $\psi_\delta(t)$ → o, *as* t → o, *for every* δ > α *and* s *is the limit as* t → o *of* θ (t) *by some Cesàro mean.*

2.2. It is convenient to give here the properties of certain functions which will be employed in the sequel. We require the functions

$$\gamma_\alpha(x),\ \overline{\gamma}_\alpha(x),\ J_\beta^\alpha(x)\ \text{and}\ \overline{J}_\beta^\alpha(x),$$

where

$$\gamma_\alpha(x) + i\overline{\gamma}_\alpha(x) = K_\alpha(x) = \int_0^1 (1 - t)^{\alpha-1} e^{itx} dt,$$

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† θ (t) → s (C) signifies that some Cesàro mean of θ (t) tends to s as t → o.

‡ I. e. $\sum \nu B_\nu = o\{n (\log n)^{2\beta+1}\}$ [R, n, α + 1, 2].

§ I. e. $\sum \nu B_\nu = o(n .\log\log n)$ [R, n, α + 1, 2].

$a > o$, and

$$J_\beta{}^\alpha (x) + i \bar{J}_\beta{}^\alpha (x) \equiv K_\beta{}^\alpha (x)$$

$$= \frac{(-1)^{h+1} x^{h+1}}{\Gamma(1+h-\alpha)\,\Gamma(\alpha+1)} \int_1^\infty (t-1)^{h-\alpha}$$

$$\times K_{1+\beta}^{(h+1)} (xt)\, dt,$$

h being the greatest integer not greater than α, and $\beta \geqslant \alpha \geqslant o$.

It is known that these functions and their derivatives are bounded for $x > o$, and that for large x

$$K_\alpha (x) = \frac{i}{x} + \Gamma(\alpha) \frac{e^{i(x - \frac{1}{2}\pi\alpha)}}{x^\alpha} + O\left(\frac{1}{x^{\alpha+1}}\right) + O\left(\frac{1}{x^2}\right),$$

while asymptotic expansions for the derivatives of $\gamma_\alpha (x)$ and $\bar{\gamma}_\alpha (x)$ are obtained by formally differentiating the above formula [3; § 1.3]. In particular it may be noted that, for λ an integer,

$$\gamma_\alpha^{(\lambda)} (x) = O\,\{(1+x)^{-\min(\alpha,\,\lambda+2)}\},$$

and

$$\bar{\gamma}_\alpha^{(\lambda)} (x) = O\,\{(1+x)^{-\min(\alpha,\,\lambda+1)}\}\ [5;\ \S\ 2\cdot 1],$$

And, for large x,

$$K_\beta{}^\alpha (x) = \frac{i}{x} + \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha)}\,\frac{e^{i[x - \pi/2(1+\beta+\alpha)]}}{x^{1+\beta-\alpha}} + O\,\frac{1}{x^{2+\beta-\alpha}} + O\left(\frac{1}{x^2}\right)$$

$$[3;\ \S\ 1.3].$$

2·3. We require a number of lemmas given below for the proofs of our theorems.

Lemma 1 [3, Theorems 1 and 3].—*For $\alpha \geqslant o$,*

$$A_\alpha (\omega) = \omega \int_0^n \phi_\alpha (u)\, J_\alpha{}^\alpha (\omega u)\, du + o\,(1),$$

as $\omega \to \infty$, where $A_\alpha (\omega)$ denotes the (R, n, α) mean of the series $\Sigma\, A_n (t)$ at $t = x$.

Lemma 2 [3, Theorems 1 c and 3a].—*For $\alpha \geqslant o$.*

$$B_\beta (\omega) = \omega \int_0^n \psi_\alpha (u)\, \{\bar{J}_\alpha{}^\alpha (\omega u) - \bar{J}^\alpha{}_{\alpha+1} (\omega u)\}\, du + o\,(1),$$

as $\omega \to \infty$, *where* $B_\alpha(\omega)$ *denotes the* (R, n, $\alpha + 1$) *mean of the sequence* $\{n \, B_n(t)\}$ *at* $t = x$, *that is to say*

$$B_\alpha(\omega) = \frac{1}{\omega^{\alpha+1}} \sum_{n < \omega} n \, B_n(\omega - n)^\alpha.$$

Lemma 3 [18; Corollary 2 to Theorem 8].—*If the series* Σc_n *be summable* (R, n, l), *for some* l, *then a necessary and sufficient condition that* Σc_n *be summable* [R, n, α, k] *is that the sequence* $\{nc_n\}$ *should be summable* [R, n, α + 1, k] *to the sum zero, that is to say*

$$\int^X \left| \frac{1}{\omega^\alpha} \sum_{n < \omega} nc_n (\omega - n)^{\alpha-1} \right|^k d\omega = o(X),$$

as $X \to \infty$.

Lemma 4.—*A necessary and sufficient condition that* $\Sigma \, B_n(t)$ *should be summable* (C) *to* s *is that* $\theta(t) \to s$ (C), *as* $t \to 0$.

Lemma 5 (See [21], § 12.42 and § 12.43).—*If* f(x) *belongs to* L^r *and* g(x) *to* $L^{p'}$, *where* $p > 1$ *and* $1/p' + 1/p' = 1$, *then*

(i) $\int |f(x) g(x)| \, dx \leqslant (\int |f(x)|^p \, dx)^{1/p} (\int |g(x)|^{p'} dx)^{1/p'}$,

and if f(x) *and* g(x) *belong to* L^p, *where* $p > 1$, *then*

(ii) $[\int |f(x) + g(x)|^p \, dx]^{1/p} < [\int |f(x)|^r \, dx]^{1/p} + [\int |g(x)|^r \, dx]^{1/p}$.

Lemma 6 [24; § 12.41].*—*If* f(x) *be a real function, periodic with period* 2π, *belonging to* L^p, $1 < p \leqslant 2$, *then*

$$[\int^e d\omega \, | \int^{2\pi} f(x) \, {\sin \omega x \atop \cos \omega x} \, dx \, |^{p'}]^{1/p'} \leqslant C \, [\int^{2\pi} |f(t)|^p \, dt]^{1/p},$$

where C *is a constant.*

Lemma 7.—

(i) $\displaystyle \int_0^\omega \left| \int_{1/\omega}^\delta g(t) \, \frac{\sin \nu t}{t} \, dt \right|^2 d\nu$

$$= O \left\{ \int_{1/\omega}^\delta \frac{g(t)}{t^2} \, dt \int_{1/\omega}^t g(u) \, \frac{\sin \omega (u - t)}{u - t} \, du \right\}$$

$$+ O \left\{ \int_{1/\omega}^\delta \frac{|g(t)|}{t^2} \, dt \int_{1/\omega}^t \frac{|g(u)|}{u} \, du \right\},$$

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* This is a modified form of the theorem given there.

and

(ii) $\displaystyle\int_0^\omega \left| \int_{1/\omega}^\delta g(t) \frac{\cos vt}{t} dt \right|^2 dv$

$$= O\left\{ \int_{1/\omega}^\delta \frac{g(t)}{t^2} dt \int_{1/\omega}^\delta g(u) \frac{\sin \omega (u-t)}{u-t} du \right\}$$

$$+ O\left\{ \int_{1/\omega}^\delta \frac{|g(t)|}{t^2} dt \int_{1/\omega}^t \frac{|g(u)|}{u} du \right\}.$$

Proof of Lemma 7.—We have

$$\int_0^\omega \left| \int_{1/\omega}^\delta g(t) \frac{\sin vt}{t} dt \right|^2 dv$$

$$= \int_{1/\omega}^\delta \int_{1/\omega}^\delta g(t) g(u) \frac{1}{tu} \int_0^\omega \sin vt \sin vu \, dv \, dt \, du$$

$$= I,$$

say. Now

$$I = \frac{1}{2} \int_{1/\omega}^\delta \int_{1/\omega}^\delta \frac{g(t) g(u)}{tu} dt \, du \int_0^\omega \{\cos v(u-t) - \cos v(u+t)\} dv$$

$$= \frac{1}{2} \int_{1/\omega}^\delta \int_{1/\omega}^\delta \frac{g(t) g(u)}{tu} \frac{\sin \omega(u-t)}{u-t} dt \, du$$

$$- \frac{1}{2} \int_{1/\omega}^\delta \int_{1/\omega}^\delta \frac{g(t) g(u)}{tu} \frac{\sin \omega(u+t)}{u+t} dt \, du$$

$$= \frac{1}{2} \{J_1 - J_2\},$$

say, and

$$J_2 = \int_{1/\omega}^\delta \frac{g(t)}{t} dt \left\{ \int_{1/\omega}^t + \int_t^\delta \right\} \frac{g(u)}{u} \frac{\sin \omega(u+t)}{u+t} du$$

$$= \int_{1/\omega}^{\delta} \frac{g(t)}{t} \, dt \int_{1/\omega}^{t} \frac{g(u)}{u} \, \frac{\sin \omega (u + t)}{u + t}$$

$$+ \int_{1/\omega}^{\delta} \frac{g(u)}{u} \, du \int_{1/\omega}^{u} \frac{g(t)}{t} \, \frac{\sin \omega (u + t)}{u + t} \, dt$$

$$= 2 \int_{1/\omega}^{\delta} \frac{g(t)}{t} \, dt \int_{1/\omega}^{t} \frac{g(u)}{u} \frac{\sin \omega (u + t)}{u + t} \, du.$$

Therefore, since

$$\frac{1}{u(u + t)} = \frac{1}{t} \left\{ \frac{1}{u} - \frac{1}{u + t} \right\},$$

$$\frac{1}{2} J_2 = \int_{1/\omega}^{\delta} \frac{g(t)}{t^2} \, dt \int_{1/\omega}^{t} g(u) \sin \omega (u + t) \left\{ \frac{1}{u} - \frac{1}{u + t} \right\} du$$

$$\leqslant \int_{1/\omega}^{\delta} \frac{|g(t)|}{t^2} \, dt \int_{1/\omega}^{t} \frac{|g(u)|}{u} \, du$$

$$+ \int_{1/\omega}^{\delta} \frac{|g(t)|}{t^2} \, dt \int_{1/\omega}^{t} \frac{|g(u)|}{u + t} \, |\sin \omega (u + t)| \, du$$

$$= O \left\{ \int_{1/\omega}^{\delta} \frac{|g(t)|}{t^2} \, dt \int_{1/\omega}^{t} \frac{|g(u)|}{u} \, du \right\}.$$

Similarly for J_1, since

$$\frac{1}{u(u - t)} = \frac{1}{t} \left(\frac{1}{u - t} - \frac{1}{u} \right),$$

$$\frac{1}{2} J_1 = \int_{1/\omega}^{\delta} \frac{g(t)}{t^2} \, dt \int_{1/\omega}^{t} g(u) \sin \omega (u - t) \left(\frac{1}{u - t} - \frac{1}{u} \right) du$$

$$\leqslant \left| \int_{1/\omega}^{\delta} \frac{g(t)}{t^2} \, dt \int_{1/\omega}^{t} g(u) \frac{\sin \omega (u - t)}{u - t} \, du \right|$$

$$+ \int_{1/\omega}^{\delta} \frac{|g(t)|}{t^2} \, dt \int_{1/\omega}^{t} \frac{|g(u)|}{u} \, du,$$

Putting together these estimates for J_1 and J_2, we get part (i) of the lemma. For part (ii) we observe that

$$\cos vt \cos vu = \tfrac{1}{2}\left[\cos v(u+t) + \cos v(u-t)\right],$$

and therefore,

$$\int_0^\omega \cos vt \cos vu \, dv = \frac{1}{2}\left(\frac{\sin \omega (u+t)}{u+t} + \frac{\sin \omega (u-t)}{u-t}\right).$$

Now the estimates for J_1 and J_2, again, give the required result.

Lemma 8 (14; p. 582, § 383).—If the periodic function $f(x)$ *be summable in* $(-\pi, \pi)$ *and* $g(x)$ *be of bounded variation in the finite interval* (α, β), *then* $\int_\alpha^\beta g(x) f(x)\,dx$ *may be evaluated by substituting for* $f(x)$ *its Fourier series, and applying term by term integration.*

Lemma 9 [19; Theorem 3].—If Σc_n *is summable* $[R, \lambda, k]$, *then* $\Sigma c_n \lambda_n^{-\delta}$, *for* $\delta > 0$, *is summable* $|R, \lambda, k|$.

Lemma 10.—If Σc_n *is summable* $[C, k]$, *then* $\Sigma c_n n^{-\delta}$, *for* $\delta > k$, *is absolutely convergent.*

Proof.—Since Σc_n is summable $[C, k]$, *i.e.*, summable $[R, n, k]$, the series $\Sigma c_n n^{-s}$, $s > 0$, is summable $|R, n, k|$ by Lemma 9, that is to say summable $|C, k|$ — because of equivalence of summabilities $|C, \alpha|$ and $|R, n, \alpha|$, [15] and hence an application of a theorem due to Kogbetliantz [16; Theorem 4] gives that $\Sigma c_n n^{-\delta}$ for $\delta > k$ is absolutely convergent.

3.1. *Proof of Theorem 1.*—For summability $[R, n, \alpha + 1, k]$ of the Fourier series (1) at $t = x$ to sum $f(x)$, by Lemma 1, it will suffice to show that

$$\int^X \left| \int_0^\eta \phi_\alpha(u) J_\alpha{}^\alpha(\omega u)\,du + o(1) \right|^k d\omega = o(X),$$

as $X \to \infty$. We write

$$\int_0^\eta \phi_\alpha(u) J_\alpha{}^\alpha(\omega u)\,du = \left(\int_0^{p/\omega} + \int_{p/\omega}^\eta\right) \phi_\alpha(u) J_\alpha{}^\alpha(\omega u)\,du = I_1(\omega) + I_2(\omega),$$

say. Then, by Lemma 5 (ii),

$$\left[\int^X \left| \omega \int \phi_\alpha(u) J_\alpha{}^\alpha(\omega u)\,du + o(1) \right|^k d\omega\right]^{1/k}$$

$$\leqslant \left[\int^X \left| \omega I_1(\omega) \right|^k d\omega\right]^{1/k} + \left[\int^X \left| \omega I_2(\omega) \right|^k d\omega\right]^{1/k} + \left[\int o(1)\,d\omega\right]^{1/k}. \quad (3.1.1)$$

Hence, it will be sufficient for our purpose to show each of the terms of the expression on the right of (3.1.1) to be

$$o(X^{1/k}).$$

Now, as $J_a{}^\alpha(\omega u)$ and $\{J_a{}^\alpha(\omega u)\}'$ are bounded, integrating by parts

$$I_1(\omega) = O\left[\int_0^{p/\omega} \phi_a(u)\, J_a{}^\alpha(\omega u)\, du\right] = \left| o(u)\, J_a{}^\alpha(\omega u)\right|_0^{p/\omega} + o\int_0^{p/\omega} u \cdot \omega du$$

$$= o\left(\frac{p}{\omega} + o\,\frac{p^2}{\omega}\right)$$

by hypothesis. Therefore

$$\int^X |\,\omega I_1(\omega)\,|\, d\omega = \int^X o(1)\, d\omega$$

$$= o(X).$$

Since, for large ωu,

$$J_a{}^\alpha(\omega u) = \frac{\sin(\omega u - \pi a)}{\omega u} + O\left(\frac{1}{(\omega u)^2}\right)$$

$$= \frac{\sin \omega u \cos \pi a}{\omega u} - \frac{\cos \omega u \sin \pi a}{\omega u} + O\left(\frac{1}{\omega^2 u^2}\right),$$

we have, by Lemma 5 (ii),

$$\left[\int^X |\,\omega I_2(\omega)\,|^k d\omega\right]^{1/k}$$

$$\leqslant \left(\int^X d\omega \left|\int_{p/\omega}^{\eta} \phi_a(u)\, \frac{\sin \omega u}{u}\, du\right|^k\right)^{1/k}$$

$$+ \left(\int^X d\omega \left|\int_{p/\omega}^{\eta} \phi_a(u)\, \frac{\cos \omega u}{u}\, du\right|^k\right)^{1/k}$$

$$+ O\left(\int^X \left|\int_{p/\omega}^{\eta} \frac{|\phi_a(u)|}{u^2}\, du\right|^k d\omega\right)^{1/k}$$

$$= I_{2,1} + \dot{I}_{2,2} + I_{2,3},$$

say. For $I_{2,3}$, we observe that, by Lemma 5 (i),

$$\int_{p/\omega}^{\eta} \frac{|\phi_a(u)|}{u^2}\, du$$

$$\leqslant \left(\int_{p/\omega}^{\eta} \frac{|\phi_a(u)|^r}{u^r}\, du\right)^{1/r} \left(\int_{p/\omega}^{\eta} \frac{du}{u^{r'}}\right)^{1/r'}$$

$$= \left(\left| \Phi_a{}^r(u) \frac{1}{u^r} \right|_{p/\omega}^{\eta} + r \int_{p/\omega}^{\eta} \frac{\Phi_a{}^r(u)}{u^r + 1} \, du \right) \mathrm{O} \left\{ \left(\frac{\omega}{p} \right)^{r'-1} \right\}^{1/r'}$$

$$= \left[\mathrm{O} \left\{ \left(\frac{\omega}{p} \right)^{r-1} \right\} + \mathrm{O} \int_{p/\omega}^{\eta} \frac{du}{u^r} \right]^{1/r} \mathrm{O} \left\{ \left(\frac{\omega}{p} \right)^{1-1/r'} \right\}$$

$$= \mathrm{O} \left[\left(\frac{\omega}{p} \right)^{r-1} \right]^{1/r} \times \mathrm{O} \left[\left(\frac{\omega}{p} \right)^{1-1/r'} \right]$$

$$= \mathrm{O} \left\{ \left(\frac{\omega}{p} \right)^{1-1/r+1-1/r'} \right\}$$

$$= \mathrm{O} \left(\frac{\omega}{p} \right).$$

Hence

$$\mathrm{I}_{2,3} = \left[\int^{X} \mathrm{O} \left(\frac{1}{p} \right) d\omega \right]^{1/k}$$

$$= \mathrm{O} \left(X^{1/k} \cdot p^{-1/k} \right) = o \left(X^{1/k} \right),$$

making $p \to \infty$.

We write

$$\int_{p/\omega}^{\eta} \phi_a(u) \frac{\sin \omega u}{u} \, du$$

$$= \int_{p/X}^{\eta} \phi_a(u) \frac{\sin \omega u}{u} \, du - \int_{p/X}^{p/\omega} \phi_a(u) \frac{\sin \omega u}{u} \, du,$$

from which we obtain

$$\mathrm{I}_{2,1} \leqslant \left(\int^{X} \left| \int_{p \cdot X}^{\eta} \phi_a(u) \frac{\sin \omega u}{u} \, du \right|^{k} d\omega \right)^{1/k}$$

$$+ \left(\int^{X} \left| \int_{p/X}^{p/\omega} \phi_a(u) \frac{\sin \omega u}{u} \, du \right|^{k} d\omega \right)^{1/k}$$

$$= \mathrm{I}'_{2,1} + \mathrm{I}''_{2,1}.$$

We first consider $I'_{2,1}$. Here we note that (i) the larger the value of k is the better is the result of our theorem [18, Theorem 4], and also (ii) the smaller the value of r is the less restrictive is the hypothesis; and therefore we suppose, without any loss of generality, that $k \geqslant 2$ and so $1 < k' \leqslant 2$ and put $r = k'$. An application of Lemma 6 gives

$$I'_{2,1} \leqslant C \left(\int_{p/X}^{\eta} \frac{|\phi_\alpha(u)|^{k'}}{u^{k'}} \, du \right)^{1/k'}$$

$$= O \left(\left| \frac{\Phi_\alpha{}^r(u)}{u^r} \right|_{p/X}^{\eta} + r \int_{p/X}^{\eta} \frac{\Phi_\alpha{}^r(u)}{u^{r+1}} \cdot du \right)^{1/r}$$

$$= O \left[\left(\frac{X}{p} \right)^{r-1} + \int_{p/X}^{\eta} \frac{1}{u^r} \cdot du \right]^{1/r}$$

$$= O \left\{ \left(\frac{X}{p} \right)^{r-1} \right\}^{1/r}$$

$$= o \left(X^{1/k} \right),$$

as $p \to \infty$ and $X \to \infty$.

In order to evaluate $I_{2,2}$, we, again, write

$$\int_{p/\omega}^{\eta} \frac{\phi_\alpha(u) \cos \omega u}{u} \, du$$

$$= \int_{p/X}^{\eta} \phi_\alpha(u) \frac{\cos \omega u}{u} \, du - \int_{p/X}^{p/\omega} \phi_\alpha(u) \frac{\cos \omega u}{u} \, du,$$

and obtain

$$I_{2,2} \leqslant \left(\int^{X} \left| \int_{p/X}^{\eta} \phi_\alpha(u) \frac{\cos \omega u}{u} \, du \right|^{k} d\omega \right)^{1/k}$$

$$+ \left(\int^{X} \left| \int_{p/X}^{p/\omega} \phi_\alpha(u) \frac{\cos \omega u}{u} \, du \right|^{k} d\omega \right)^{1/k}$$

$$= I'_{2,2} + I''_{2,2}.$$

And, an application of Lemma 6 gives

$$I'_{2,2} = o \left(X^{1/k} \right).$$

Now it remains to show that

$$I''_{2,1} = o(X^{1/k}),$$

or

$$(I''_{2,1})^k = o(X), \tag{3.1.2}$$

and

$$(I''_{2,2})^k = o(X). \tag{3.1.3}$$

On integration by parts, it is easy to see that

$$\int_{p/X}^{p/\omega} \phi_\alpha(u) \frac{\exp(i\omega u)}{u}\, du$$

$$= \left| o(u) \left| \frac{\exp(i\omega u)}{u} \right| \right|_{p/X}^{p/\omega} + o \int_{p/X}^{p/\omega} u \left| \exp(i\omega u) \left(\frac{1}{u^2} - \frac{i\omega}{u} \right) \right|\, du$$

$$= o(1) + o\left(\log \frac{X}{\omega} \right) + o(p)$$

$$= o(1) + o\left(\log \frac{X}{\omega} \right). \tag{3.1.4}$$

Hence in order that (3.1.2) and (3.1.3) may hold, it is sufficient to prove that

$$\int^X \left(\log \frac{X}{\omega} \right)^k d\omega = O(X).$$

For this by Hölder's inequality, *viz.*, Lemma 5 (i), it suffices to show that, for any integral l,

$$J = \int^X \left(\log \frac{X}{\omega} \right)^l d\omega = O(X).$$

And by repeated integrations by parts, we obtain

$$J = \int^X \left(\log \frac{X}{\omega} \right)^l d\omega$$

$$= O\left[\left| \omega \left(\log \frac{X}{\omega} \right)^l \right|^X + 1 \int^X \omega \left(\log \frac{X}{\omega} \right)^{l-1} \frac{1}{\omega}\, d\omega \right]$$

$$= \ldots\ldots$$

$$= \ldots\ldots$$

$$= O \left[\left| \omega \left(\log \frac{X}{\omega} \right)^l \right|^X + \left| l\omega \left(\log \frac{X}{\omega} \right)^{l-1} \right|^X \right.$$

$$\left. + \cdots + l! \left| \omega \log \frac{X}{\omega} \right|^X + l! \int^X d\omega \right]$$

$$= O(X).$$

Thus the theorem is proved.

3.2. *Proof of Theorem 2.*—Here, as in Theorem 1, it is to be shown that

$$I = \int^X \left| \omega \int_0^\eta \phi_\alpha (u) J_\alpha{}^\alpha (\omega u) \, du + o(1) \right|^2 d\omega = o(X),$$

as $X \to \infty$. And, by Lemma 5 (ii),

$$I^{1/2} \leqslant \left[\int^X \left| \omega \int_0^{1/\omega} \phi_\alpha (u) J_\alpha{}^\alpha (\omega u) \, du \right|^2 d\omega \right]^{1/2}$$

$$+ \left(\int^X \left| \omega \int_{1/\omega}^\eta \phi_\alpha (u) J_\alpha{}^\alpha (u\omega) \, du \right|^2 d\omega \right)^{1/2} + \left(\int^X o(1) \, d\omega \right)^{1/2}$$

$$= I_1{}^{1/2} + I_2{}^{1/2} + I_3{}^{1/2},$$

say. Evidently

$$I_3 = o(X).$$

Also, since $J_\alpha{}^\alpha (x)$ is bounded and under the given condition

$$\int_0^t |\phi_\alpha (u)| \, du = o(t),$$

we get

$$I_1 \leqslant \int_0^X \omega^2 \cdot O \left[\int_0^{1/\omega} |\phi_\alpha (u)| \, du \right]^2 d\omega$$

$$= O \int_0^X \omega^2 \, o \left(\frac{1}{\omega^2} \right) d\omega$$

$$= o(X).$$

Now, for ωu large

$$J_\alpha{}^\alpha (\omega u) = \frac{\sin (\omega u - \pi \alpha)}{\omega u} + O \left(\frac{1}{\omega^2 u^2} \right),$$

therefore

$$I_2{}^{1/2} \leqslant \left(\int^X \left| \omega \int_{1/\omega}^\eta \phi_\alpha (u) \frac{\sin (\omega u - \pi \alpha)}{\omega u} \, du \right|^2 d\omega \right)^{1/2}$$

$$+ O\left(\int\limits_{1/\omega}^{x}\left|\int\limits^{\eta}\left|\phi_\alpha(u)\right|\cdot\frac{1}{\omega u^2}\cdot du\right|^2 d\omega\right)^{1/2}$$

$$\leqslant\left(\int\limits_{1/X}^{x}\left|\omega\int\limits^{1/\omega}\phi_\alpha(u)\frac{\sin\omega u}{\omega u}du\right|^2 d\omega\right)^{1/2}$$

$$+\left(\int\limits_{1/X}^{x}\left|\omega\int\limits^{\eta}\phi_\alpha(u)\frac{\sin\omega u}{\omega u}du\right|^2 d\omega\right)^{1/2}$$

$$+\left(\int\limits_{1/X}^{x}\left|\omega\int\limits^{1/\omega}\phi_\alpha(u)\frac{\cos\omega u}{\omega u}du\right|^2 d\omega\right)^{1/2}$$

$$+\left(\int\limits_{1/X}^{x}\left|\omega\int\limits^{\eta}\phi_\alpha(u)\frac{\cos\omega u}{\omega u}du\right|^2 d\omega\right)^{1/2}$$

$$+ O\left(\int\limits_{1/\omega}^{x}\left|\int\limits^{\eta}\left|\phi_\alpha(u)\right|\frac{1}{\omega u^2}du\right|^2 d\omega\right)^{1/2}$$

$$= I_{2,1}^{1/2} + I_{2,2}^{1/2} + I_{2,3}^{1/2} + I_{2,4}^{1/2} + I_{2,5}^{1/2},$$

say. And

$$I_{2,5} = \int\limits_0^x \frac{1}{\omega^2}\, d\omega\left(\int\limits_{1/\omega}^{\eta}\frac{|\phi_\alpha(u)|}{u^2}\,du\right)^2.$$

But

$$\int\limits_{1/\omega}^{\eta}\frac{|\phi_\alpha(u)|}{u^2}\cdot du = \left|\, o(u)\cdot\frac{1}{u^2}\,\right|_{1/\omega}^{\eta} + o\int\limits_{1/\omega}^{\eta}\frac{u}{u^3}\cdot du$$

$$= o(\omega) + o(\omega) = o(\omega).$$

Hence

$$I_{2,5} = \int\limits_0^x \frac{1}{\omega^2}\cdot o(\omega^2)\, d\omega = o(X).$$

Also

$$I_{2,1} \leqslant \int\limits_0^x\left|\,\omega\int\limits_{1/X}^{1/\omega}|\phi_\alpha(u)|\,du\,\right|^2 d\omega = \int\limits_0^x o(1)\, d\omega = o(X),$$

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For $I_{2,2}$, we have by Lemma 7 (i)

$$I_{2,2} = O \left(\int\limits_{1/X}^{\eta} \frac{\phi_\alpha(t)}{t^2} \cdot dt \int\limits_{1/X}^{t} \phi_\alpha(u) \frac{\sin X(u-t)}{u-t} \, du \right)$$

$$+ O \left(\int\limits_{1/X}^{\eta} \frac{|\phi_\alpha(t)|}{t^2} \cdot dt \int\limits_{1/X}^{t} \frac{|\phi_\alpha(u)|}{u} \cdot du \right) \tag{3.2.1}$$

It follows from the hypothesis in the Theorem that

$$\int\limits_{1/X}^{t} \phi_\alpha(u) \frac{\sin X(u-t)}{u-t} \, du = O \left(X \int\limits_{0}^{t} |\phi_\alpha(u)| \, du \right)$$

$$= o \left[\frac{Xt}{(\log 1/t)^\beta} \right] .$$

Hence

$$\int\limits_{1/X}^{\eta} \frac{\phi_\alpha(t)}{t^2} \cdot dt \int\limits_{1/X}^{t} \phi_\alpha(u) \frac{\sin X(u-t)}{u-t} \, du$$

$$= o \left[X \int\limits_{1/X}^{\eta} \frac{|\phi_\alpha(t)|}{t} \left(\log \frac{1}{t} \right)^{-\beta} \cdot dt \right] . \tag{3.2.2}$$

Writing

$$\Phi(t) = \int\limits_{0}^{t} |\phi_\alpha(u)| \, du,$$

we have

$$\int\limits_{1/X}^{\eta} \frac{|\phi_\alpha(t)|}{t} \left(\log \frac{1}{t} \right)^{-\beta} \cdot dt$$

$$= \left| \frac{\Phi(t)}{t} \left(\log \frac{1}{t} \right)^{-\beta} \right|_{1/X}^{\eta} + \int\limits_{1/X}^{\eta} \frac{\Phi(t)}{t^2} \left(\log \frac{1}{t} \right)^{-\beta} \cdot dt$$

$$- \beta \int\limits_{1/X}^{\eta} \frac{\Phi(t)}{t^2} \left(\log \frac{1}{t} \right)^{\beta-1} \cdot dt$$

$$= O(1) + o \int\limits_{1/X}^{\eta} \frac{1}{t} \left(\log \frac{1}{t} \right)^{-2\beta} dt + o \int\limits_{1/X}^{\eta} \frac{1}{t} \left(\log \frac{1}{t} \right)^{-2\beta-1} \cdot dt$$

$$= O(1), \tag{3.2.3}$$

since $\beta > 1/2$. Also

$$\int\limits_{1/X}^{t} \frac{|\phi_\alpha(u)|}{u} \cdot du = \left| o(u) \cdot \frac{1}{u} \right|_{1/X}^{t} + \int\limits_{1/X}^{t} \frac{o(u)}{u^2} \cdot du$$

$$= o(\log Xt)$$

Therefore

$$\int\limits_{1/X}^{\eta} \frac{|\phi_\alpha(t)|}{t^2} \cdot dt \int\limits_{1/X}^{t} \frac{|\phi_\alpha(u)|}{u} \cdot du$$

$$= o \int\limits_{1/X}^{\eta} \frac{|\phi_\alpha(t)|}{t^2} \cdot \log Xt \cdot dt$$

$$= \left| o(t) \log \frac{Xt}{t^2} \right|_{1/X}^{\eta} + o \int\limits_{1/X}^{\eta} \frac{t}{t^2} \cdot dt + o \int\limits_{1/X}^{\eta} \frac{t}{t^3} \cdot \log(Xt)\,dt$$

$$= o(X) + o(\log X) + o \int\limits_{1}^{\eta X} X \frac{\log u}{u^2} \cdot du$$

$$= o(X). \tag{3.2.4}$$

From (3.2.1), (3.2.2), (3.2.3) and (3.2.4) follows that

$$I_{2,2} = o(X).$$

That

$$I_{2,4} = o(X),$$

follows from an application of Lemma 7 (ii) exactly on the lines of the above. We have now to show that

$$I_{2,3} = \int\limits^{X} \left| \omega \int\limits_{1/X}^{1/\omega} \phi_\alpha(u) \frac{\cos \omega u}{\omega u} \cdot du \right|^2 d\omega = o(X),$$

which follows from (3.1.4).

This completes the proof of the Theorem.

3.3. *Proof of Theorem 3.*—In this case we have to show

$$\int\limits^{X} \left| \omega \int\limits^{\eta} \phi_\alpha(u) J_\alpha^{\alpha'}(\omega u)\,du + o(1) \right|^2 d\omega = o \begin{cases} X(\log X)^{2\beta+1}, & \beta > -\frac{1}{2} \\ X \log \log X, & \beta = -\frac{1}{2}, \end{cases}$$

and this simply reduces to the consideration of

$$I = \int\limits_{}^{X} |\; \omega \int\limits_{0}^{\eta} \phi_a(u)\, J_a{}^a(\omega u)\, du\;|^2\, d\omega.$$

Splitting the inner integral, *i.e.*, putting

$$\int\limits_{0}^{\eta} = \int\limits_{0}^{1/\omega} + \int\limits_{1/\omega}^{\eta},$$

we get

$$(I^{1/2}) \leqslant (\int\limits_{}^{X} |\; \omega \int\limits_{0}^{1/\omega} \phi_a(u)\, J_a{}^a(\omega u)\, du\;|^2\, d\omega)^{1/2}$$

$$+ [\int\limits_{}^{X} |\; \omega \int\limits_{1/\omega}^{\eta} \phi_a(u)\, J_a{}^a(\omega u)\, du\;|^2\, d\omega]^{1/2}$$

$$= (I_1)^{1/2} + (I_2)^{1/2},$$

say. Now, since $J_a{}^a(\omega u)\, du$ is bounded,

$$\int\limits_{0}^{1/\omega} \phi_a(u)\, J_a{}^a(\omega u)\, du = O \int\limits_{0}^{1/\omega} |\, \phi_a(u)\, |\, du$$

$$= o\left[\frac{(\log \omega)^\beta}{\omega}\right],$$

by hypothesis, and therefore

$$I_1 = o \int\limits_{}^{X} (\log \omega)^{2\beta}\, d\omega = o \int\limits_{}^{X} \frac{\omega\, (\log \omega)^{2\beta}}{\omega} \cdot d\omega$$

$$= o\left(X \int\limits_{}^{X} \frac{(\log \omega)^{2\beta}}{\omega} \cdot d\omega \right)$$

$$= o \begin{cases} X\, (\log X)^{2\beta+1}, \text{ for } \beta > -\tfrac{1}{2} \\ X \log \log X, \text{ for } \beta = -\tfrac{1}{2}. \end{cases}$$

For large ωu,

$$J_a{}^a(\omega u) = \frac{\sin(\omega u - \pi a)}{\omega u} + O\left(\frac{1}{\omega^2 u^2}\right).$$

Hence

$$(I_2)^{1/2} \leqslant \left[\int\limits_{}^{X} \left(\omega \int\limits_{1/\omega}^{\eta} \phi_a(u)\, \frac{\sin \omega u}{\omega u}\, du \right)^2 d\omega \right]^{1/2}$$

$$+ \left[\int_{1/\omega}^{X} \left(\omega \int^{\eta} \phi_a(u) \frac{\cos \omega u}{\omega u} du \right)^2 d\omega \right]^{1/2}$$

$$+ \left[\int_{1/\omega}^{X} \left(\omega \int^{\eta} \frac{|\phi_a(u)|}{\omega^2 u^2} du \right)^2 d\omega \right]^{1/2}$$

$$= (I_{2,1})^{1/2} + (I_{2,2})^{1/2} + (I_{2,3})^{1/2}.$$

Also

$$\int_{1/\omega}^{\eta} \frac{|\phi_a(u)|}{u^2} \cdot du = \left| o \left\{ u \left(\log \frac{1}{u} \right)^{\beta} \right\} \frac{1}{u^2} \right|_{1/\omega}^{\eta}$$

$$+ \int_{1/\omega}^{\eta} \frac{o \left\{ u (\log 1/u)^{\beta} \right\}}{u^3} \cdot du$$

$$= o \left\{ \omega (\log \omega)^{\beta} + o \int_{1/\omega}^{\eta} \frac{(\log 1/u)^{\beta-1}}{u^2} \cdot du \right.$$

$$= o \left\{ \omega (\log \omega)^{\beta} \right\} + o \left\{ \omega (\log \omega)^{\beta-1} \log \omega \right\}$$

$$= o \left\{ \omega (\log \omega)^{\beta} \right\}. \tag{3.3.1}$$

Therefore

$$I_{2,3} = o \int^{X} (\log \omega)^{2\beta} d\omega$$

$$= o \begin{cases} X (\log X)^{2\beta+1}, & \text{for } \beta > -\tfrac{1}{2} \\ X \log \log X, & \text{for } \beta = -\tfrac{1}{2}. \end{cases}$$

Now

$$I_{2,1} = \int^{X} \left(\omega \int_{1/\omega}^{\eta} \frac{\sin \omega u}{\omega u} \phi_a(u) \, du \right)^2 d\omega$$

$$= \int^{X} \left(\omega \int_{1/X}^{\eta} \frac{\sin \omega u}{\omega u} \phi_a(u) \, du - \omega \int_{1/X}^{1/\omega} \frac{\sin \omega u}{\omega u} \phi_a(u) \, du \right)^2 d\omega$$

$$= O \int^{X} \left(\omega \int_{1/X}^{\eta} \frac{\sin \omega u}{\omega u} \phi_a(u) \, du \right)^2 d\omega$$

$$- + O \int_{1/X}^{X} \left(\omega \int^{1/\omega} \frac{\sin \omega u}{\omega u} \phi_a(u) \, du \right)^2 d\omega$$

$$= O (I'_{2,1}) + (I''_{2,1}),$$

say. Again

$$I''_{2,1} = \int_{1/X}^{X} \left(\omega \int_{1/X}^{1/\omega} \frac{\sin \omega u}{u} \phi_\alpha(u)\, du \right)^2 d\omega$$

$$\leqslant \int_{1/X}^{X} \left[\int_{1/X}^{1/\omega} | \phi_\alpha(u) |\, du^2 \right] d\omega$$

$$= o \int^{X} (\log \omega)^{2\beta} d\omega$$

$$= o \begin{cases} X (\log X)^{2\beta+1}, & \text{for } \beta > -\frac{1}{2} \\ X \log \log X, & \text{for } \beta = -\frac{1}{2}. \end{cases}$$

And, by Lemma 7 (i),

$$I'_{2,1} = \int_{1/X}^{X} \left| \int_{1/X}^{\eta} \frac{\sin \omega u}{u} \phi_\alpha(u)\, du \right|^2 d\omega$$

$$= O \int_{1/X}^{\eta} \frac{\phi_\alpha(t)}{t^2}\, dt \int_{1/X}^{t} \phi_\alpha(u) \frac{\sin X(u-t)}{u-t}\, du$$

$$+ O \int_{1/X}^{\eta} \frac{| \phi_\alpha(t) |}{t^2}\, dt \int_{1/X}^{t} \frac{| \phi_\alpha(u) |}{u}\, du$$

$$= O (J_1) + O (J_2),$$

say. We have

$$\int_{1/X}^{t} | \phi_\alpha(u) | \frac{du}{u}$$

$$= \left| o \left\{ u \left(\log \frac{1}{u} \right)^\beta \right\} \frac{1}{u} \right|_{1/X}^{t} + o \int_{1/X}^{t} u \left(\log \frac{1}{u} \right)^\beta \frac{du}{u^2}$$

$$= O (1) + o \{(\log X)^\beta\} + o \{(\log X)^{\beta+1}\}$$

$$= o \{(\log X)^{\beta+1}\}.$$

Therefore from (3.3.1) follows that

$$J_2 = o \{X (\log X)^{2\beta+1}\}.$$

Now, for J_1, we have

$$\int_{1/X}^{t} \phi_\alpha(u) \frac{\sin X(u-t)}{u-t}\, du \leqslant X \int_{1/X}^{t} | \phi_\alpha(u) |\, du$$

$$= o\left\{ \mathrm{X}t \left(\log \frac{1}{t}\right)^{\beta}\right\}.$$

Hence

$$\mathrm{J}_1 = o\int_{1/\mathrm{X}}^{\eta} \frac{\mathrm{X}\,|\,\phi_\alpha(t)\,|}{t}\left(\log\frac{1}{t}\right)^{\beta} dt$$

$$= o\left[\mathrm{X}\left\{\left|\,o\left(t\left(\log\frac{1}{t}\right)^{\beta}\right)\frac{\left(\log\frac{1}{t}\right)^{\beta}}{t}\right|_{1/\mathrm{X}}^{\eta}\right.\right.$$

$$\left.\left.+\,o\int_{1/\mathrm{X}}^{\eta} t\left(\log\frac{1}{t}\right)^{\beta}\left(\frac{\left(\log\frac{1}{t}\right)^{\beta}}{t^2}+\beta\frac{\left(\log\frac{1}{t}\right)^{\beta-1}}{t^2}\right)dt\right\}\right]$$

$$= o\left[\mathrm{X}\left\{\mathrm{O}\,(1)+o\,(\log \mathrm{X})^{2\beta}+o\int_{1/\mathrm{X}}^{\eta}\frac{\left(\log\frac{1}{t}\right)^{2\beta}}{t}dt\right.\right.$$

$$\left.\left.+\,o\int_{1/\mathrm{X}}^{\eta}\frac{\left(\log\frac{1}{t}\right)^{2\beta-1}}{t^2}\right\}\right]$$

$$= o\left\{ \begin{array}{l} \mathrm{X}\,(\log \mathrm{X})^{2\beta+1},\ \text{for}\ \beta > -\tfrac{1}{2}\\ \mathrm{X}\log\log \mathrm{X},\ \text{for}\ \beta = -\tfrac{1}{2}. \end{array}\right.$$

We have now only to show that

$$\mathrm{I}_{2,2} = o\left\{ \begin{array}{l} \mathrm{X}\,(\log \mathrm{X})^{2\beta+1},\ \text{for}\ \beta > -\tfrac{1}{2}\\ \mathrm{X}\log\log \mathrm{X},\ \text{for}\ \beta = -\tfrac{1}{2}. \end{array}\right.$$

Again, we break up $\mathrm{I}_{2,2}$ as follows

$$\mathrm{I}_{2,2} = \int^{\mathrm{X}}\left\{\omega\int_{1/\mathrm{X}}^{\eta}\frac{\cos\omega u}{u}\phi_\alpha(u)\,du - \omega\int_{1/\mathrm{X}}^{1/\omega}\frac{\cos\omega u}{u}\phi_\alpha(u)\,du\right\}^2 d\omega$$

$$= \mathrm{O}\left[\int^{\mathrm{X}}\left\{\omega\int_{1/\mathrm{X}}^{\eta}\frac{\cos\omega u}{u}\phi_\alpha(u)\,du\right\}^2 d\omega\right.$$

$$\left.+\int^{\mathrm{X}}\left\{\omega\int_{1/\mathrm{X}}^{1/\omega}\frac{\cos\omega u}{u}\phi_\alpha(u)\,du\right\}^2 d\omega\right]$$

$$= \mathrm{O}\,(\mathrm{I'}_{2,2})+\mathrm{O}\,(\mathrm{I''}_{2,2})$$

That $I'_{2,2}$ satisfies the required order condition follows on the lines of the proof given for $I'_{2,1}$. Also

$$\int_{1/X}^{1/\omega} \cos \omega u \, \frac{\phi_a(u)}{u} \, du$$

$$\leqslant \int_{1/X}^{1/\omega} \frac{|\phi_a(u)|}{u} \, du$$

$$= \left| o \, u \left(\log \frac{1}{u} \right)^\beta \frac{1}{u} \right|_{1/X}^{1/\omega} + \int_{1/X}^{1/\omega} \frac{o \left\{ u \left(\log \frac{1}{u} \right)^\beta \right\}}{u^2} \, du$$

$$= o \{ (\log X)^\beta \} + o \{ (\log \omega)^\beta \} + o \{ (\log X)^{\beta+1} - (\log \omega)^{\beta+1} \}.$$

Hence

$$I''_{2,2} = o \int^X (\log X)^{2\beta} d\omega + o \int^X (\log \omega)^{2\beta} d\omega + o \int^X \{ (\log X)^{\beta+1}$$

$$- (\log \omega)^{\beta+1} \}^2 \, d\omega.$$

Integrating twice by parts, the last integral, we get

$$I''_{2,2} = o \{ X (\log X)^{2\beta} \} + o \int^X (\log \omega)^{2\beta} d\omega + o \{ (\log X)^{2\beta+2} \}$$

$$+ o \{ (\log X)^{2\beta+1} \} + o \int^X (\log X)^{\beta+1} (\log \omega)^{\beta-1} \, d\omega$$

$$+ o \int^X (\log \omega)^{2\beta} d\omega$$

$$= o \begin{cases} X (\log X)^{2\beta+1}, \text{ for } \beta > -\tfrac{1}{2} \\ X \log \log X, \text{ for } \beta = -\tfrac{1}{2}. \end{cases}$$

And this completes the proof.

3.4. *Proofs of Theorems 4, 5 and 6.*—In case of Theorems 4 and 5, by Lemmas 2, 3 and 4 and the hypothesis on $\theta(t)$ common in both of the theorems, it is only to be shown that

$$\int^X \left| \omega \int_0^{\eta} \psi_a(t) \left[\overline{J}_a{}^\alpha(t\omega) - \overline{J}_{a+1}{}^\alpha(\omega t) \right] dt + o(1) \right|^l d\omega = o(X),$$

as $X \to \infty$, where $l = k$ or 2, according as we consider Theorem 4 or Theorem 5.

For ωt large, we can write

$$\overline{J}_a{}^\alpha(\omega t) - \overline{J}_{a+1}{}^\alpha(\omega t) = \overline{J}_a{}^\alpha(\omega t) - \frac{1}{\omega t} - \left(\overline{J}_{a+1}{}^\alpha(\omega t) - \frac{1}{\omega t} \right).$$

With these considerations, Theorems 4 and 5 follow exactly by the procedure of Theorems 1 and 2. Theorem 6 is to be proved in the like manner on the lines of Theorem 3.

It may be remarked here that if in the hypothesis of these theorems o is replaced by O the analogous results with boundedness and O in the conclusion are obtained. Only in case of Theorems 2 and 4, the given conditions with o and O come out to be equivalent.

3·5. *Proof of Theorem 7.*—Since the Fourier series of $\phi(t)$ is given by

$$\Sigma A_n(x) \cos nt†$$

and $(1-u)^{\delta-1}$ is of bounded variation in $(o, 1-\epsilon)$, where $o < \epsilon < 1$, we have, by Lemma 8,

$$\phi_{\delta,\epsilon}(t) = \delta \int_0^{1-\epsilon} (1-u)^{\delta-1} \phi(tu)\, du$$

$$= \delta \Sigma A_n \int_0^{1-\epsilon} (1-u)^{\delta-1} \cos ntu\, du$$

$$= \delta \Sigma A_n \gamma_{\delta,\epsilon}(nt),$$

where

$$\gamma_{\delta,\epsilon}(nt) = \int_0^{1-\epsilon} (1-u)^{\delta-1} \cos ntu\, du.$$

If $\delta > 1$ the same is true with $\epsilon = o$, and we therefore have

$$\phi_\delta(t) = \delta \Sigma A_n [\gamma_\delta(nt)]. \tag{3.5.1}$$

If $o < \delta < 1$, it may be easily shown that

$$\gamma_\delta(x) - \gamma_{\delta,\epsilon}(x) = \begin{cases} O(\epsilon^\delta), & \text{for } x \geqslant o \\ O(x^{-\delta}), & \text{for large values of } x. \end{cases}$$

Hence, for every positive t,

$$\lim_{\epsilon \to o} \Sigma A_n [\gamma_\delta(nt) - \gamma_{\delta,\epsilon}(nt)]$$

$$= \lim_{\epsilon \to o} (O \sum_0^N |A_n|\, \epsilon^\delta + O \sum_{N+1}^\infty |A_n|\, n^{-\delta}).$$

Now, since ΣA_n is summable $[R, n, \alpha]$, the series $\Sigma |A_n| n^{-\delta}$, $\delta > \alpha$, is convergent by Lemma 10 and therefore, for $o < \alpha < \delta < 1$,

$$\lim_{\epsilon \to o} \phi_{\delta,\epsilon}(t) = \delta \Sigma A_n \gamma_\delta(nt).$$

Thus (3.5.1) is true for all $\delta > \alpha$, provided the integral $\phi_\delta(t)$ is interpreted in the Cauchy sense. Hence, by (3.5.1),

---

† We take $f(x) = o$, for convenience, without any loss of generality.

$$\phi_\delta(t) = \sum_0^\infty A_n \gamma_\delta(nt)$$

$$= \sum_0^\infty A^\bullet(n) \{\gamma_\delta(nt) - \gamma_\delta(n+1)t\}$$

$$= - t \sum A^o(n) \int_0^{n+1} \gamma'_\delta(tu) \, du$$

$$= - t \int_0^\infty \gamma'_\delta(tu) A(u) \, du.$$

Integrating by parts h times, where h is the greatest integer less than α, we obtain

$$\int_0^\infty \gamma'_\delta(ut) A(u) \, du = | \sum_{v=1}^h (-1)^{v-1} [\Gamma(v+1)^{-1} A^v(u) \gamma_\delta^{(v)}(ut) t^{v-1}] |_{u=0}^\infty$$

$$+ \frac{(-1)^v t^h}{\Gamma(h+1)} \int_0^\infty \gamma_\delta^{h+1}(ut) A^h(u) \, du.$$

Now $A^v(o) = o$, and for every fixed positive t, as $u \to \infty$,

$$A^v(u) \gamma_\delta^{(v)}(ut) = o\{u^{\min(v+1,\,\alpha)}\} O\{u^{-\min(\delta,\,v+2)}\} = o(1),$$

where $v = 1, 2, \cdots, h$. Since, because of summability (R, n, α) of the Fourier series (1) at $t = x$ to $f(x) = o$.

$$A^\alpha(u) = o(u^\alpha),$$

therefore, for $v < \alpha$,

$$A^v(u) = o(u^\alpha) \text{ [13, Theorem 22]}, \ddagger$$

and, also,

$$A_n(x) = o(1),$$

therefore

$$A^v(x) = o(x^{v+1}),$$

for all v. Hence the integrated terms all vanish and

$$\phi_\delta(t) = \frac{(-1)^{h+1} t^{h+1}}{\Gamma(h+1)} \int_0^\infty \gamma_\delta^{(h+1)}(ut) A^h(u) \, du.$$

Now, in case α is an integer

$$\phi_\delta(t) = \frac{(-1)^\alpha t^\alpha}{\Gamma(\alpha)} \int_0^\infty \gamma_\delta^{(\alpha)}(ut) A^{\alpha-1}(u) \, du.$$

---

‡ $A^v(u)$ stands for $C_n^v(u)$ of the Theorem.

Put $t = 1/w$ in the above, then

$$\phi_\delta\left(\frac{1}{w}\right) = \frac{(-1)^a}{\Gamma(a)\cdot w^a} \int_0^\infty \gamma_\delta^{(a)}\left(\frac{u}{w}\right) A^{a-1}(u)\, du$$

$$= \frac{(-1)^a}{w^a\, \Gamma(a)} \int_0^\infty \gamma_\delta^{(a)}(v)\, A^{a-1}(wv)\, w\, dv$$

$$= \frac{(-1)^a}{w^a\, \Gamma(a+1)} \int_0^\infty \gamma_\delta^{(a)}(v)\, dA^a(wv)$$

$$= \frac{(-1)^a}{w^a\, \Gamma(a+1)} \left(\int_0^\eta + \int_\eta^\infty\right) \gamma_\delta^{(a)}(v)\, dA^a(wv)$$

$$= I_1 + I_2,$$

say. Now, as can be easily seen,

$$I_1 = o\,(1),$$

as $w \to \infty$. And

$$I_2 \leqslant \frac{K}{w^a} \cdot \int_\eta^\infty |\, dA^a(wv)\,|\, v^{-\delta}\, \S$$

$$= \left|\frac{o\,(wv)^a v^{-\delta}}{w^a}\right|_\eta^\infty + \frac{1}{w^a}\, o \int_\eta^\infty w^a v^{a-1-\delta} dv$$

$$= o\,(1),$$

since $\delta > a$.

For non-integral a, we have

$$\phi_\delta(t) = \frac{(-1)^{h+1} t^{h+1}}{\Gamma(h+1)} \int_0^\infty \gamma_\delta^{h+1}(ut)\, A^h(u)\, du$$

$$= K t^{h+1} \int_0^\infty \gamma_\delta^{(h+1)}(ut)\, du \int_0^u (u-v)^{h-a} dA^a(v) \quad [\text{see}^{13}\, \text{p.27, Lemma 6}]$$

$$= K t^{h+1} \int_0^\infty dA^a(v) \int_v^\infty (u-v)^{h-a} \gamma_\delta^{(h+1)}(ut)\, du,$$

---

§ K denotes a constant, which may be different at different places.

provided the inversion of the order of integration is justified. To show that this is so, it will be sufficient to prove that, as $X \to \infty$,

$$I = \int_0^X dA^\alpha (v) \int_X^\infty (u - v)^{h-\alpha} \gamma_\delta^{(h+1)} (ut)\, du \to o,$$

for every fixed positive t. Write

$$\int_X^\infty (u - v)^{h-\alpha} \gamma_\delta^{(h+1)} (ut)\, du = (\int_X^{X+1} + \int_{X+1}^\infty) (u - v)^{h-\alpha} \gamma_\delta^{(h+1)} (ut)\, du$$

$$= M_1 + M_2.$$

Now, supposing $\delta < h + 2$,

$$M_1 = O \int_X^{X+1} (u - v)^{h-\alpha} u^{-\delta} du = O\, (X^{-\delta}),$$

and by the second mean value theorem

$$M_2 = (X + 1 - v)^{h-\alpha} \int_{X+1}^\mu \gamma_\delta^{(h+1)} (tu)\, du, \; X + 1 < \mu < \infty$$

$$= O\, (X^{-\delta})$$

Therefore

$$I = O\, [X^{-\delta} \int_0^X | dA^\alpha (u) |\,] = O\, (X^{-\delta}) \cdot o\, (X^\alpha) = o\, (1),$$

since (1) is summable $[R, n, \alpha]$ [18; § 4]. By a similar method it can be shown that

$$\gamma_\delta^{(\alpha)} (v) \equiv \int_v^\infty (u - v)^{h-\alpha} \gamma_\delta^{(h+1)} (u)\, du = O\, (v^{-\delta}).$$

Put $t = 1/w$, then

$$\phi_\delta \left(\frac{1}{w} \right) = Kw^{-(h+1)} \int_0^\infty dA^\alpha (v) \int_v^\infty (u - v)^{h-\alpha} \gamma_\delta^{(h+1)} \left(\frac{u}{w} \right) du$$

$$= Kw^{-\alpha} \int_0^\infty dA^\alpha (wv) \int_0^\infty (u - v)^{h-\alpha} \gamma_\delta^{(h+1)} (u)\, du$$

$$= Kw^{-\alpha} \int_0^\infty \gamma_\delta^{(\alpha)} (v)\, dA^\alpha (wv).$$

Now

$$\phi_\delta (t) \to o,$$

as $t \to o$, or

$$\phi_\delta \left(\frac{1}{w} \right) \to o,$$

as $w \to \infty$, follows as in case of integral a,

Thus the theorem is completely proved.

3.6.   *Proof of Theorem 8.*—Since the Fourier series of $\psi(t)$ is given by $\Sigma B^n(x) \sin nt$, an analysis similar to the proof of Theorem 7 shows that in case of summability $[R, n, \alpha]$ of the conjugate series $\psi_\delta(t) \to o$, as $t \to o$; and by Lemma 4 it follows that s is the limit, as $t \to o$, of

$$\int_{to}^{\infty} \frac{\psi(u)}{u} \, du$$

by some Cesàro mean.

REFERENCES

1.   BOSANQUET, L. S., 1930, "On the summability of Fourier series," *Proc. London Math. Soc.*, **31** (2), 144–64.

2.   ——————, 1932, "On strongly summable Fourier series, *Jour. London Math. Soc.*, **7**, 47–52.

3.   ——————, 1934, "On the Cesàro summation of Fourier series and allied series, *ibid.*, **37** (2), 17–32.

4.   ——————, 1936, "Arithmetic means and Fourier series," *Trans. American Math. Soc.*, **39**, 189–204.

5.   —————— AND HYSLOP, J. M., 1937, "On the absolute summability of allied series of a Fourier series," *Math. Zeitschrift.*, **42**, 489–512.

6.   BOYD, A. V. AND HYSLOP, J. M., 1952, "A definition for strong Rieszian summability and its relationship to strong Cesàro's summability," *Proc. Glasgow Math. Assoc.*, **1**, 34–39.

7.   CARLEMAN, T., 1923, "A theorem concerning Fourier series," *Proc. London Math. Soc.*, **21** (2), 483–92.

8.   MIN-TEH CHENG, 1947, "Summability factors of Fourier series at a given point," *Duke Math. Jour.*, **14**, 405–10.

9.   HARDY, G. H. AND LITTLEWOOD, J. E., 1913, "Sur la sèrie de Forier d'nue fonction a carré sommable," *Comptes Rendus*, **156**, 1307–09.

10.   ——————, 1926, "The allied series of a Fourier series," *Proc. London Math. Soc.*, **24** (2), 211–46.

11.   ——————, 1927, "Notes on the theory of series (iv). On the strong summability of Fourier series," *ibid.*, **26** (2), 273–86.

12.   ——————, 1935, "The strong summability of Fourier series," *Fundamenta Mathematicæ*, **25**, 162–89.

13.   HARDI, G. H. AND RIESZ, M., 1915, *The General Theory of Dirichlet's Series*, Cambridge.

14.   HOBSON, E. W., 1927, *The Theory of Functions of a Real Variable*, Cambridge.

15. HYSLOP, J. M., 1936, "The absolute summability of series by Rieszian means," *Proc. Edinburgh Math. Soc.*, **5**, 46–54.

16. KOGBETLIANTZ, E., 1925, "Sur les séries absolument sommables par la méthode des moyennes arithmétiques," *Bulletin des Sciences Mathématiques,* **49**, (2), 234–56.

17. SINGH, U. N., 1947, "On the strong summability of a Fourier series and its conjugate series," *Proc. National Institute of Sciences of India*, A, **13**, 319–25.

18. SRIVASTAVA, P., 1957, "On strong Rieszian summability of infinite series," *ibid.*, **23**, 58–71.

19. ——————, 1956, "On strong Rieszian summability of Dirichlet's series," *Proc. Indian Science Congress Association, Abstracts*, p. 7.

20. SUTTON, O. G., 1925, On a theorem of Carleman," *Proc. London Math. Soc.*, **23**, 48–51.

21. TITCHMARSH, E. C., 1939, *The Theory of Functions*, Oxford.

22. WANG, F. T., 1944, "Note on the H_2 summability of Fourier series," *Jour. London Math. Soc.*, **19**, 208–09.

23. WINN, C. E., 1933, "On strong summability for any positive order," *Math. Zeitschrift*, **37**, 481–92.

24. ZYGMUND, A., 1935, *Trigonometrical Series*, Warsaw.

SOME OBSERVATIONS REGARDING THE RELATION BETWEEN THE SPECIFIC HEAT AND THE THERMAL CONDUCTIVITY OF METALS (SOLIDS) AT ORDINARY TEMPERATURE

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ABSTRACT

Depending on the practical values, the relation has been obtained purely from a graphical method by arranging the metals into subgroups according to periodic classification. The general equation for a particular subgroup is found to be $S = aK^2 + bK + c$. The curves of subgroups *a* are concave *upwards* and those of subgroups *b* are concave *downwards*. There is a distinct and *regular* variation of the constants *a*, *b* and *c* with subgroups *a* or *b* and so also the axes of the different curves. Finally, Dulong and Petit's law has been modified from the idea of atom structure and has been applied to obtain the equation $Z \times K^2 = a$ constant, for a particular sub-group which follows the practical values. The new formula has been applied to find out the thermal conductivity of metals whose values were not available during the work. The actual values of the thermal conductivity of some of these metals were obtained later on and were found to agree very well with the predicted values.

INTRODUCTION

It is well known that in the case of gases, according to the kinetic theory, there is a definite theoretical relation between K and C_v, which, in general, is found to be supported by experimental results. On the other hand, although considerable work has been done separately on the theory of specific heat of metals and the theory of thermal conductivity, there does not seem to be any co-relation established between the two based on theoretical considerations. Before making any such attempts, it will be fruitful to examine the available data and establish an empirical relation, if any, between K and C_v for metals (solids).

In this paper, an attempt has been made in this direction.

EXPERIMENTAL DATA

The data for different metals have been given in Table I. The specific heat S and the thermal conductivity K are expressed in Cal./gm./° C. and Cal./cm.²/cm./° C./sec. respectively at 20° C.

OBSERVATIONS

Any theoretical explanation will have to take into account the peripheral electronic structure of the atom and a proper method of approach will be to

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TABLE I

Atomic No. Z	Element	S	K	S × Z
3	Li	0·79	0·17	2·370
4	Be	0·428	0·392	..
11	Na	0·295	0·323	3·245
12	Mg	0·249	0·370	2·988
13	Al	0·215	0·521	2·795
14	Si	0·176	0·20	2·464
15	P	0·177	..	2·655
16	S	0·175	..	2·800
19	K	0·177	0·24	3·363
20	Ca	0·157	..	3·140
22	Ti	0·125	..	2·750
23	V	0·120	..	2·760
24	Cr	0·107	0·16	2·568
25	Mn	0·114	..	2·850
26	Fe	0·112	0·19	2·812
27	Co	0·099	0·16	2·673
28	Ni	0·105	0·22	2·940
29	Cu	0·092	0·941	2·668
30	Zn	0·093	0·26	2·790
31	Ga	0·079	..	2·449
32	Ge	0·073	..	2·336
33	As	0·082	..	2·706
34	Se	0·075	..	2·550
37	Rb	0·080	..	2·960
38	Sr	0·176	..	..
40	Zr	0·066	..	2·640
42	Mo	0·061	0·349	2·562
44	Ru	0·056	..	2·464
45	Rh	0·059	0·21	2·655

TABLE I (*Contd.*)

Atomic No. Z	Element	S	K	S × Z
46	Pd	0·054	0·17	2·484
47	Ag	0·056	0·975	2·632
48	Cd	0·055	0·22	2·640
49	In	0·057	0·057	2·793
50	Sn	0·054	0·153	2·700
51	Sb	0·049	0·045	2·499
52	Te	0·047	0·014	2·444
53	I	0·052	..	2·756
55	Cs	0·052	..	2·860
56	Ba	0·068	..	..
58	Ce	0·042	..	2·436
59	Pr	0·048	..	2·832
60	Nd	0·045	..	2·700
73	Ta	0·033	0·13	2·409
74	W	0·032	0·475	2·368
75	Re	0·033	..	2·475
76	Os	0·031	..	2·346
77	Ir	0·032	0·14	2·464
78	Pt	0·032	0·16	2·496
79	Au	0·031	0·707	2·449
80	Hg	0·033	0·0201	2·640
81	Tl	0·031	0·093	2·511
82	Pb	0·031	0·084	2·542
83	Bi	0·034	0·0201	2·822
90	Th	0·034	...	3·060
92	U	0·028	0·064	2·576

The mean value of S × Z = 2.671.

examine the elements group by group according to the usual periodic classification.

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So, keeping in view that the properties of the metals will follow, more or less, the same periodicity, the metals have been grouped accordingly as subgroups 1a, 1b, 2a, 2b, 3a, 3b, 4a, 4b, 5a, 5b, 6a, 6b, 7a, 7b exactly as in the periodic table and the transition elements separately as 8_1 containing Fe, Co, Ni, and 8_2 containing Ru, Rh, Pd and 8_3 containing Os, Ir, Pt.

For metals belonging to a subgroup, the values of S are plotted against the corresponding values of K and the nature of the curve so formed is examined. When the points are extrapolated carefully, it is found that the characteristic curve for each subgroup is a parabola.

For the sake of clarity a few of the typical curves are shown in Fig. 1.
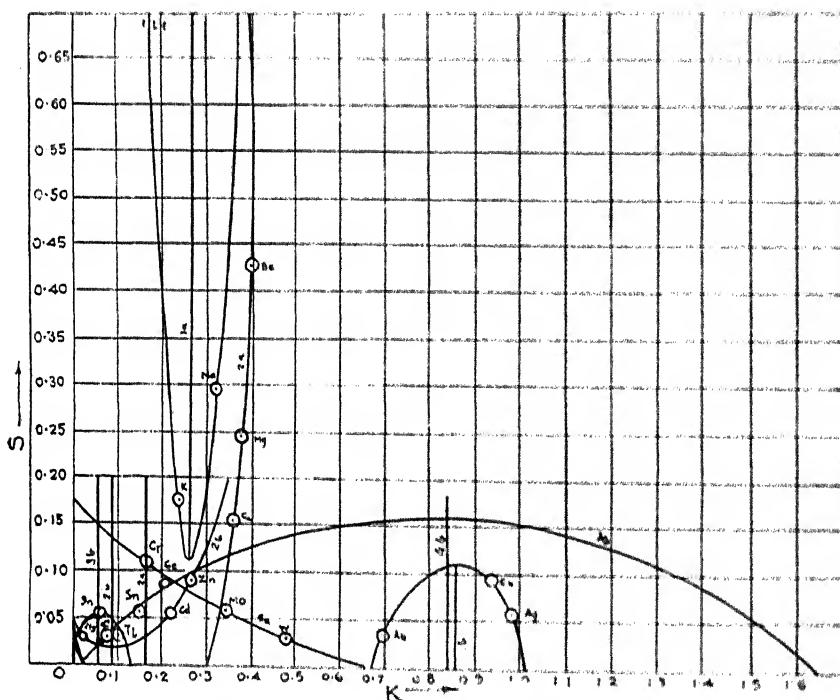
FIG. 1. Showing the relations between S and K of different subgroups of periodic table.

A close examination of these curves brings out the following points:

(1) The curve for each subgroup is a parabola.

(2) The axes of the parabolas and their positions are different for different subgroups.

(3) The curves of subgroups *a* are concave upwards and those of subgroups *b* are concave downwards. The only exception to this being 2b which includes mercury which is a liquid at ordinary temperature.

(4) So far as the transition elements are concerned, group 8_1 gives curve like those of subgroups *a* while groups 8_2 and 8_3 give curves like those of subgroups *b*.

GENERAL DISCUSSION

All the different parabolas of different subgroups can be represented by the general form

$$S = aK^2 + bK + c, \tag{1}$$

where a, b and c are constants. According to eq. (1), the vertex of the corresponding curve will be the point $\left(-\dfrac{b}{2a}, \; c - \dfrac{b^2}{4a}\right)$ and the focus of it will be the point $(0, 1/4a)$ from the vertex which determines the concavity of the curve.

In Tables II and III are shown the values of these constants for different subgroups with the exceptions of subgroups 3a, 4a, 5a, 6b, 7a and 7b where sufficient data are not available to form the equations.

TABLE II

For curves of Fig. 1 having concavities in the upward directions

Sub-groups No.	a	b	c	b^2	$4ac$	$b^2 - 4ac$	$4a^2$	$\dfrac{b^2 - 4ac}{4a^2}$
1a	66·5	−36·1	5·01	1303·2	1332·6	−29·4	17689	−0·0017
2a	16·12	− 5·29	*	27·98	*	27·98	1039·4	0·027
(2b)	2·77	− 0·53	0·043	0·281	0·476	− 0·195	30·68	−0·0063
6a	0·042	− 0·26	0·147	0·0676	0·0247	0·0429	0·0069	6·2
8$_1$	100	−34·56	3·07	1194·4	1228	−33·6	40000	−0·0008

TABLE III

For curves of Fig. 1 having concavities in the downward directions

Subgroups No.	a	b	c	b^2	$4ac$	$b^2 - 4ac$	$4a^2$	$\dfrac{b^2 - 4ac}{4a^2}$
1b	− 4·53	7·72	−3·17	59·6	57·4	2·2	82·08	0·027
3b	−18·51	2·055	*	4·22	*	4·22	1370·5	0·003
4b	− 0·24	0·39	*	0·152	*	0·152	0·23	0·66
5b	−24·4	2·19	*	4·796	*	4·796	2381·4	0·002
8$_2$	− 0·916	0·473	*	0·224	*	0·224	0·3884	0·577
8$_3$	− 1·428	0·426	*	0·181	*	0·181	8·156	0·022

* Nearly negligible (in both Tables II and III).

According to the Tables II and III, if the different curves are again plotted for *a*, *b* and *c* respectively against subgroup numbers, it will be seen that there are some regular variations of *a*, *b* and *c* with the subgroups. These have been shown in Fig. 2 and Fig. 3 respectively for the Tables II and III.
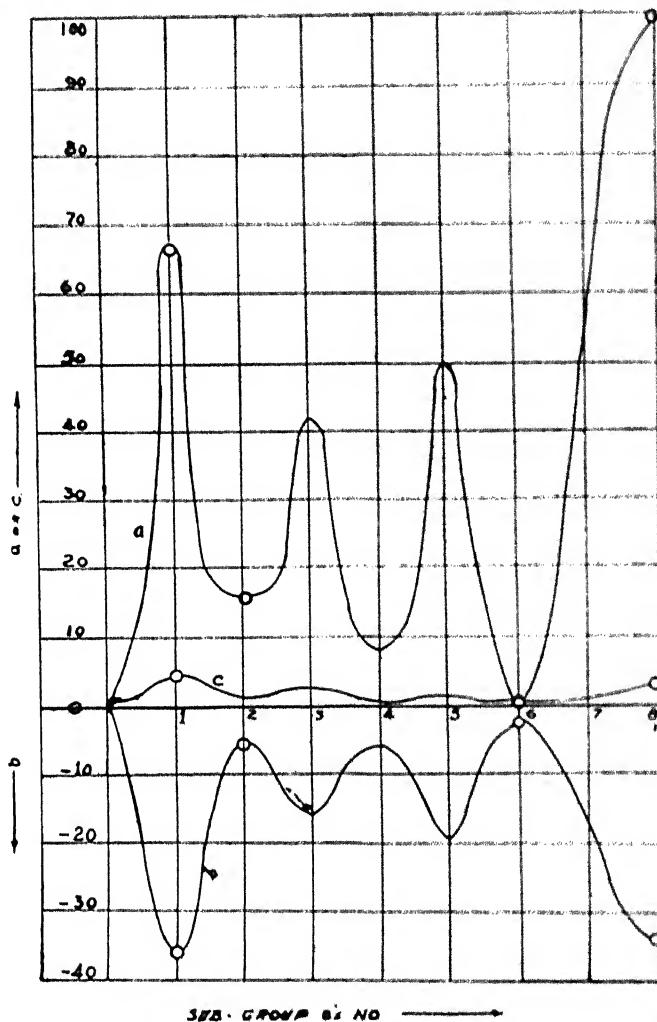
FIG. 2. Showing the relation of the variations of *a*, *b* or *c*. with the subgroups corresponding to the curves of Fig. 1 having concavities in the upward directions.

The curves have been extrapolated to keep the symmetry throughout where the points are not properly obtained due to lack of sufficient data. These regular variations in the constants may give a possible support for the empirical relation between the specific heat and the thermal conductivity of solids at the ordinary temperature obtained earlier.

It has been observed further that there is also a good relation between the shifting of the axes of the curves of Fig. 1 of different subgroups and the sub-
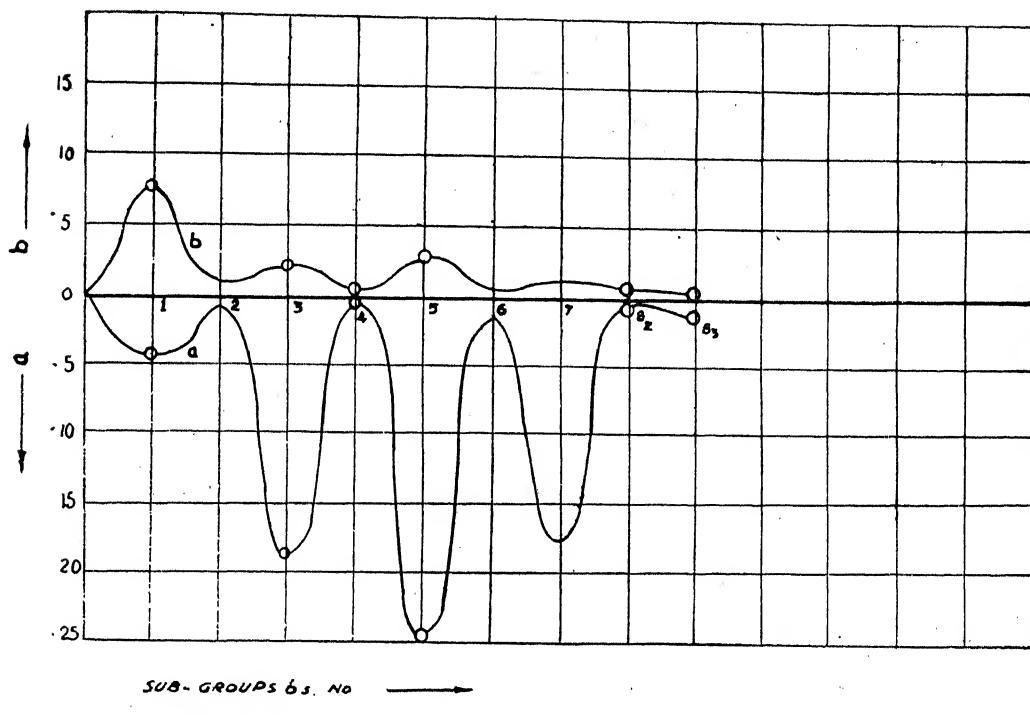
FIG. 3. Showing the relations of the variations of *a* and *b* with the subgroups corresponding to the curves of Fig. 1 having concavities in the downward directions.

group numbers. The relation has been shown in Fig. 4 in accordance with the Tables IV and V.

TABLE IV

For curves of Fig. 1 having concavities in the upward directions

Subgroups No.	Axis position
1a	0·26
2a	0·16
(2b)	0·09
6a	3·20
8_1	0·17

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TABLE V

For curves of Fig. 1 having concavities in the downward directions

| Subgroups No.	Axis position
1b	0·86
3b	0·06
4b	0·85
5b	0·05
8₂	0·25
8₃	0·15
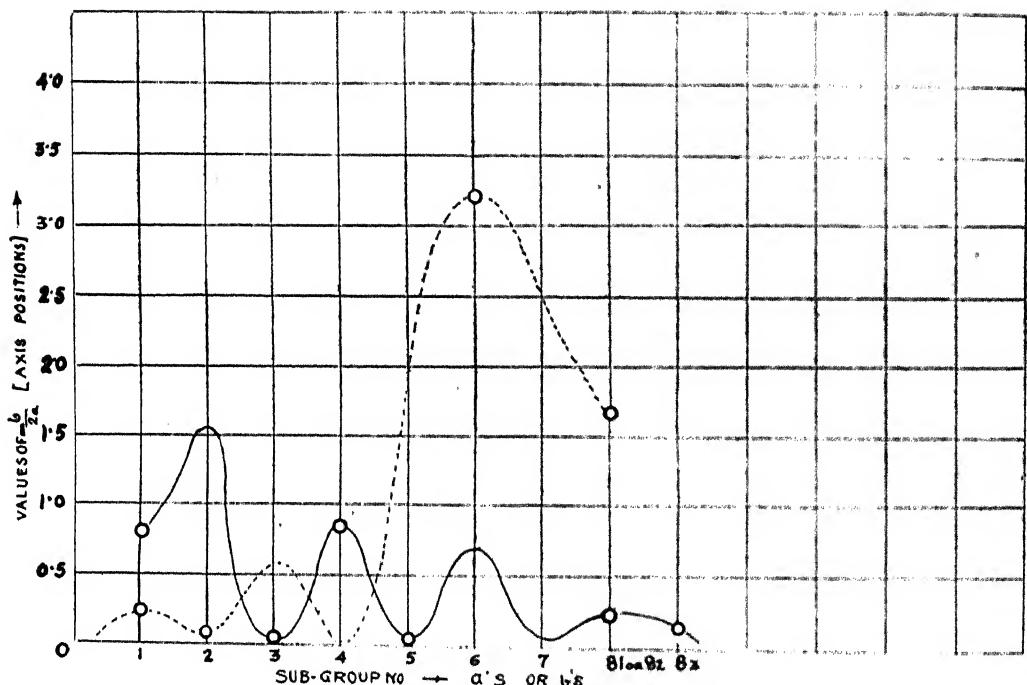
FIG. 4. Showing the relation between the axis positions and the subgroups corresponding to curves of Fig. 1. The thick line corresponds to the subgroups *b*'s and the dotted line corresponds to the subgroups *a*'s.

The relation with subgroups *a* could not be obtained properly due to lack of sufficient data. The points for subgroups *b* are plotted and extrapolated to

keep the symmetry throughout and it has been interpreted that the shifting of the axis is oscillating in nature from group to group in the increasing order and the oscillation is like a damped one having the amplitudes decreasing in order as the subgroup number increases. According to the few points obtained, the nature of the curve of the subgroups *a* may be interpreted similarly to be just the opposite to that of subgroups *b*. That is to say, the amplitudes of oscillation will go on increasing as the subgroup number increases.

Figures 2, 3 and 4 may give a good basis to form a theoretical explanation of the results. The meaning of the constants *a*, *b* and *c* with their fixed signs for different subgroups may also be found from the above-mentioned figures.

MODIFIED DULONG PETIT'S LAW

According to Dulong and Petit's law of atomic heat of solids at ordinary temperature, it is shown that

$$\text{Atomic weight} \times \text{Specific heat} = 6 \text{ (approx.)}$$
$$= \text{a constant.}$$

In any periodic property of the elements the atomic number plays a more important role than the atomic weight. Hence it will be worthwhile to examine Dulong and Petit's law taking Z (atomic number) instead of A (atomic weight), *i.e.*,

$$Z \times S = \text{a constant} = d \text{ (say)} \tag{2}$$

The mean value of this constant of the new relation has been calculated and found to be 2·671 with extremes of 3·363 and 2·336. Fig. 5 shows the plot between S and Z drawn from the data given in Table I.
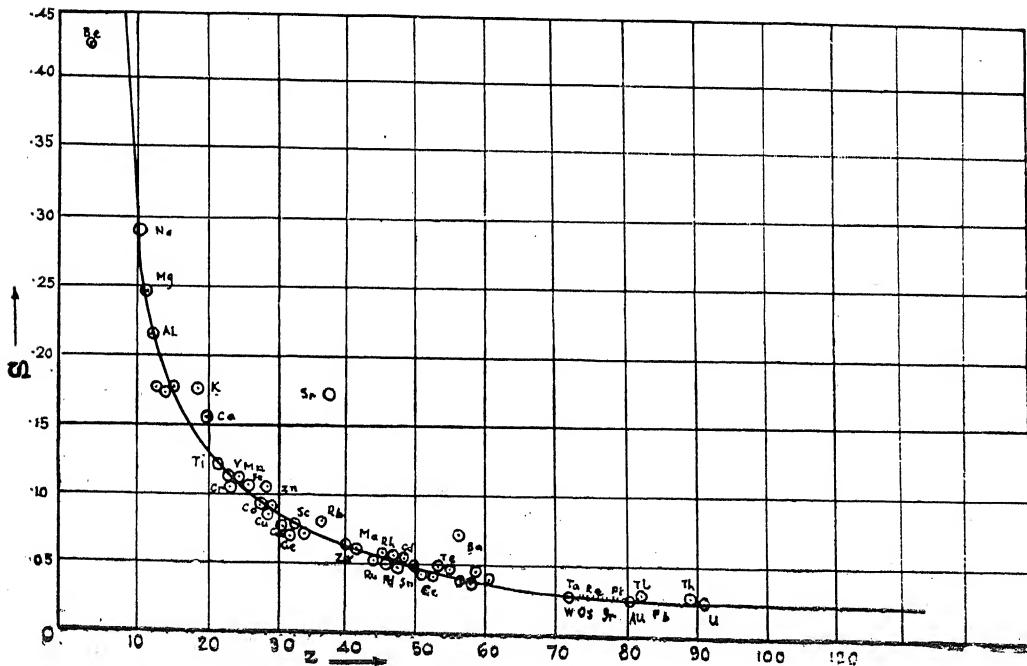
FIG. 5. Showing the relation between S and Z according to Table I.

All the metals except a few lie on the mean curve, thus showing a greater degree of accuracy than the Dulong and Petit's law.

From equations (1) and (2) we get

$$Z \left(K + \frac{b}{2a} \right)^2 = \frac{d}{a} \qquad (3)$$

when $b^2 \simeq 4ac$ or $(b^2 - 4ac)/4a^2$ becomes a very small quantity as shown in Tables II and III.

Further if the axis of Z shifts to $(-b/2a, 0)$, equation (3) reduces to

$$Z \times K^2 = \frac{d}{a} = \text{a constant} \qquad (4)$$

for a particular subgroup. The equation (4) will then give the nature of the curve as shown in Fig. 6.
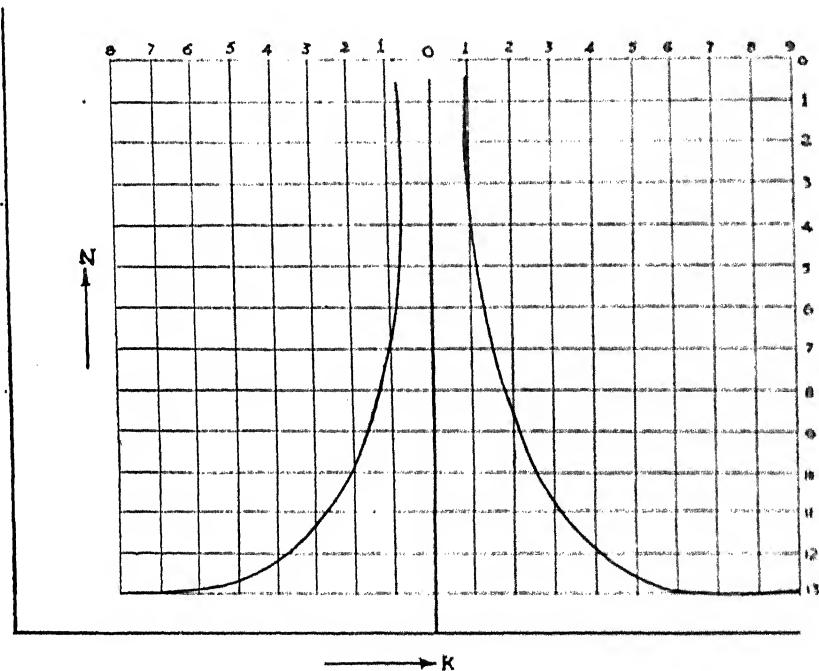
FIG. 6. Showing the relation between Z and K drawn according to Eq. (4) deduced theoretically.

The curves drawn between Z and K from the data given in Table I for different subgroups are shown in Fig. 7. It is to be noted that these are exactly similar to the theoretical curve in Fig. 6.

APPLICATIONS

It will be interesting to use the results of the above discussion and the equations and graphs for obtaining the values of the thermal conductivity of certain
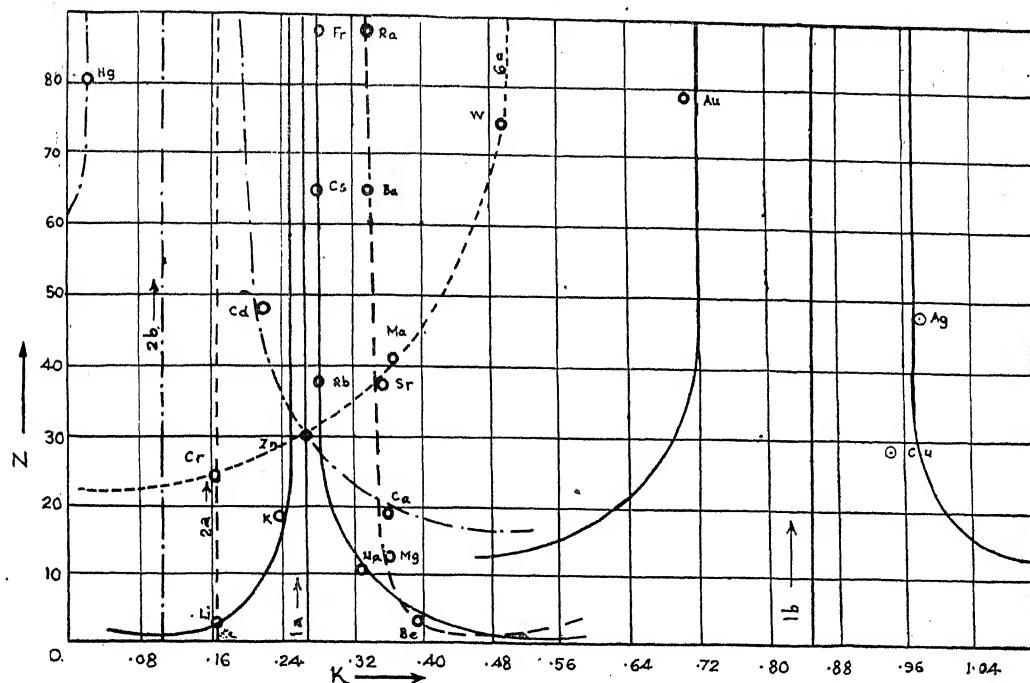
FIG. 7. Showing the relations between Z and K for different subgroups from the experimental data as given in Table I.

metals for which experimental data are not available. The following list gives the values for K so calculated.

$$†Ca = 0·35, \quad †Ga = 0·06, \quad †Ge = 0·21, \quad As = 0·045,$$
$$Rb = 0·27, \quad †Sr = 0·36, \quad Ru = 0·18, \quad Cs = 0·27,$$
$$Ba = 0·34, \quad Os = 0·12, \quad Fr = 0·268, \quad Ra = 0·334.$$

SUMMARY

It is interesting to note that the empirical relation between the specific heat and the thermal conductivity of metals at ordinary temperature for a particular subgroup has been found to be $S = aK^2 + bK + c$, the properties of which are

(1) The curve for each subgroup is a parabola.

(2) The axes of the parabolas and their positions are different for different subgroups.

(3) The curves of subgroups *a* are concave upwards and those of sub-groups *b* are concave downwards. The only exception to this being 2*b* which includes mercury which is a liquid at ordinary temperature.

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† The actual values for K for these metals have subsequently been obtained and were found to agree very well with the calculated values given,

(4) The curve of one transition element group is exactly like that of subgroup *a* while the two other transition element groups give curves like those of subgroups *b*.

(5) There is a distinct and regular variation of the constants *a*, *b* and *c* with subgroups *a* or *b* and so also the axes of the different curves.

(6) The empirical relation also suggests a modified form of Dulong and Petit's law which finally gives the equation $Z \times K^2 =$ a constant, for a particular subgroup which follows the practical values.

(7) Finally the new formula can be applied to find out the thermal conductivity of metals whose values have not been determined experimentally so far.

REFERENCES

1. SAHA, M. N. AND SRIVASTAVA, B. N., 1950, *A Treatise on Heat*. Third edition.

2. *Circular of the National Bureau of Standards*, 1943. C. 447, p. 459.

3. TOMKEIEFF, S. I., 1954, *A New Periodic Table of the Elements*.

4. HUBBARD, H. D. AND MEGGERS, W. F., 1947, *Periodic Chart of the Atoms*.

A STUDY OF THE RELATIONSHIP BETWEEN THE SIZE OF THE KIDNEY AND THE SIZE OF THE FISH

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A DETAILED investigation of the various metabolic processes taking place in the animal body and of enzymic systems responsible for these processes, not only gives us a peep into the mechanism of the subdued biochemical reactions but it also gives us an idea of the various pathways that the different metabolic processes followed during their evolution from the simple functions of serving as steps in the fundamental process of life, *viz.*, duplication, self-energy release and adaptation, to a vital footstone in the struggle of the animal through the prolonged evolution of life. It is only the acclimatisation of these enzymic systems to the changing physico-chemical changes of the environment and their adaptability to the milder changes with changed chemical behaviour which is responsible for the evolution of the whole living unit in general. The changed enzymic property by mild changing physico-chemical conditions was followed by the changed chemical property[1] and the formation of changed end products[2] out of which those which made the organism more suited for the environment were retained and in due course underwent further evolution and those which were unhelpful slowly perished. The external organs were formed to protect and help these enzymic systems to perform these activities and thus the bony system, muscle construction or even the structures of the various organs of an animal are not the fundamentals of evolution but just the outer coat or the extra accessory organs to protect the vital enzymic system which remains the fundamental driving agency of the evolution.

Here a study of the nitrogen excretory organ, the kidney, of several fishes has been carried out and it has been observed that considering the animals of about the same habits, the animals with higher body weight have lower number of glomeruli per kg. wt. of the body as compared to the animals with the lower body weight. With lesser number of glomeruli the metabolites concentration in the body increases and some of these metabolites activate the proteinases activity and increase the protein formation in the animal body and this is followed by the increase in size of the animals resulting in the subsequent development of the supporting and other accessory organs during the course of evolution for the better functioning of the big body.

Table II indicates that the fishes (3) of smaller size have larger number of glomeruli per kg. wt. of their body weight as compared to the fishes of bigger size. Thus increased number of glomeruli which means larger excretions of nitrogenous and mineral products makes the system of the organism poor in these ingredients and thus the system remains small. However when the excretion of nitrogenous matter and minerals by the organism becomes small the organs remain rich in these substances and the fixed nitrogen, carbon and minerals which are

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TABLE I

Showing number of glomeruli in a few fresh-water fishes of different sizes

Species	Family	Body wt. by gram	No. of glomeruli in one kidney	No. of glomeruli in one kidney per kg. body wt.	Body surface area sq. cm.	No. of glomeruli in one kidney per sq. cm. body surfase
Fresh-water Fishes:						
1 *Epinephelus striatus*	Serranidæ	896	19,044	21,254	837	22·8
2 *Anguilla rostrata*	Anguillideæ	765	17,082	22,303	753·3	22·6
3 *Gadus callarias Linnæus*	Gadidæ	670	16,250 (Rt.)	24,000	688·5	23·6
4 *Cryptacanthodes maculatus*	Cryptacanthodidæ	560	6,402	11,428	702	9·1
5 *Ocyurus chrysurus*	Lutianidæ	448	8,837	19,725	526·9	16·8
6 *Pseudopleuronectes americanus*	Pleuronectidæ	160	5,301	33,131	265·5	20·1
7 *Caranx ruber* (Block)	Carangidæ	90	13,298	147,753	180·9	73·5
8 *Ameiurus nebulosus*	Siluridæ	88·5	18,160 (L.)	205,197	179·4	101·2
9 *Diplodus argenteus*	Sparidæ	30	4,488	149,333	86·9	51·6
10 *Sarinella umbitoria*	Clupeidæ	25	1,890	75,600	76·9	24·6
11 *Plecostemus commersoni* (valenciennes)	Loricariidæ	3·3	1,000 (Rt.)	304,545	20	50·3

TABLE II

Indicating the number of glomeruli in a few Elasmobranchs and Cyclostomes of different sizes

Species	Family	Body wt. by gram	No. of glomeruli in one kidney	No. of glomeruli in one kidney per kg. body wt.	Body surface area sq. cm.	No. of glomeruli in one kidney per sq. cm. body surface
Elasmobranchs:						
1 *Rajadiaphanes Mitchill*	Rajidæ	1,740	1,816 (Rt.)	1,043	1300·5	·803
2 *Rajaerinacea Mitchill*	"	1,060	1,197 (Rt.)	1,129	934·2	1·20
3 *Mustelus canis*	Galeidæ	485	4,397 (Rt.)	8,939	555·3	7·74
Cyclostomes:						
1 *Myxine glutinosa*	Myxinidæ	250	34 (Rt.)	136	357·3	·095
2 "	"	225	40 (Rt.) 37 (L.)	172	333	·115
3 "	"	210	39 (Rt.) 38 (L.)	156	317·8	·122
4 "	"	140	44 (Rt.) 43 (L.)	186	242·8	·177

the vital constituents of the tissues and the body and the enzymically guided metabolic processes are so adjusted that they are utilised by them for the construction of body matter and the tissues multiply. As these excretory materials cannot be reutilised for the formation of the body matter their formation has to be retarded by better and efficient metabolic processes. It is interesting to consider whether it is the decreased number of glomeruli which affected the evolution of the metabolic processes with better and efficient mechanism leading to the decreased formation of the excretory products or the metabolic changes were affected first in the course of evolution leading to lesser formation of excretory matter and a consequent decrease in the number of glomeruli.

If molecular evolution theory for the origin of life is supposed to be correct then the molecular mechanism of metabolic processes has to be considered as affected first by the processes of evolution. This is possible if it is considered that the intake of newer molecule by the organism for their synthesis in the animal body has taken place.

The lesser excretion of inorganic and organic substances from the body of the organism increases their concentration inside the body and thus increases the interferences of these foreign bodies with the enzymic systems of the tissue. It has been observed that a foreign body in an enzymic reaction profoundly influences the reaction both quantitatively and qualitatively. The proteins in general are extremely susceptible to form a complex with a number of organic and inorganic materials and these complexes show a great variation in their enzymic properties from the original protein. Mentioning about the changes of enzymic properties of the protein complexes[4–9] Deborin in his report of the International Symposium on Origin of Life held at Moscow in 1957 mentions that the formation of complexes by primordial proteins with one another and with other organic substances could have given rise to an enormous variety of compounds and chemical processes which in the subsequent evolutions could have produced various conjugated proteins possessing specific biochemical functions and could have engendered the whole set of enzymic processes essential to life.

It appears that some of the inorganic and organic substances formed either by the degradation taking place in animal body or assimilated directly by the alimentary canal help in the formation of larger quantity of body protein which is the chief constituent of the animal body and they may improve the reactions of the metabolism in such a way that the lesser material to be thrown out as by-products are formed. These reactions will obviously be accompanied by the increase in weight of the animal and the other subsidiary parts of the animal body are formed for the better functioning of the enlarged body.

So there is always a decreased number of glomeruli per kg. body wt. in the fishes of larger size. It appears that the activity of enzymes by increased concentration of foreign body is chiefly by the accumulation of the excretory products in the system providing greater changes of activator—substrate complex rather than their enrichment due to oral intake, because there remains a basic selection of the organism of eating only a few types of things and not eating a few other things and so the concentration of some such activator in one and not in the other organisms is not very probable. Of course with better and efficient mechanism of protein formation the body demand of food will be greater and the organism which is bigger will consume larger quantity of food.

REFERENCES

1. SINHA, R. C., 1957, *Doctoral Thesis*, Allahabad.

2. DEBORIN. G. A., 1957, " The Origin of Life on the Earth (Moscow)," **25 A**, 258.

3. NASH, J., 1931, *Am. J. Anat.*, **47** (2), 425-45.

4. WAUGH, D. F., 1954, *Ad. post. Chem.*, **9**, 326.

5. CHEPINOGU, O. P., 1956, *Igd. AN, USSR Kiew*.

6. CHARGAFF, M. ZIFF AND MOORE, D. H. 1941, *J. Biol. Chem.*, **139**, 383.

7. DERIVCHIAN, D. F., 1949, *Discs. Far. Soc.*, **6**, 7.

8. PUTNAM, F., 1948, *Adv. Prot. Chem.*, **4**, 80.

9. KLOTZ, J., 1953, *The Proteins*, **1**, New York.

PRODUCTION OF ALKALINITY IN NORMAL SOILS WITH SPECIAL REFERENCE TO THE FORMATION OF POTASSIUM SOILS

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IT is well known that Mondesir[1] was the pioneer to convert a normal soil into an alkaline soil by treating it twenty times with a solution of sodium chloride followed by distilled water. Gedroiz,[2] de Sigmond,[3] Dominicis,[4] Prescott,[5] Cummins and Kelley,[6] Burgess[7] and others corroborated the work of Mondesir.

According to Ivanova,[8] Prasolov and Antepov-Karataev[9] and Pankov and Shavrygin,[10] the percentage of sodium in the exchange-complex of the solonetz need not be higher than 5–10%. Arany and Kelley[11] are of the opinion that 19% is the minimum and 75% is the maximum sodium content of the exchange-complex of solonetz whilst Remezov[12] finds no such high percentage of sodium.

Kelley[13] describes that the total adsorbed sodium is not necessarily correlated with pH. More or less sodium will be adsorbed by exchange with the hydrogen ions of hydroxyl group, therefore, alkali soils which are strongly alkaline chemically may not necessarily be sodium saturated. Misra[14] has observed that sodium adsorbed from very dilute solutions of sodium carbonate is not able to hydrolyse and produce any alkalinity in the soil after washing it with distilled water.

So far no one has reported studies on soils in which exchangeable potassium has been considered to be the chief cause of alkalinity though theoretically potassium ions my behave as sodium ions. Hence the present investigations were taken up by treating the normal soils with different potassium salts.

EXPERIMENTAL METHODS

Two soils designated as normal and garden soils were selected for the present investigations. Their chemical composition and their exchangeable bases were determined by the hydrochloric acid extract and neutral normal ammonium acetate method respectively after drying the soil samples, previously passed through a 70 mesh sieve, in an oven at 105° C. for 4–6 hours.

100 gm. of the soils were treated with equimolar quantities of chlorides and sulphates of sodium and potassium, sodium carbonate and sodium bicarbonate. Their combinations with calcium carbonate and calcium chloride were also added to the normal soil. The mixtures were kept in enamelled dishes, moistened daily with distilled water and exposed daily to sunlight for 8 hours. This process of wetting and drying in the sun continued for a month, accompanied with daily stirring the mixture with glass rods. The determinations of pH and the exchangeable bases were then carried out. For the determination of the pH values of the pastes (1 : 2·5 soil-water ratio) or of leachates, the Beckman-pH-meter was used.

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The exchangeable calcium, sodium and potassium were determined in the ammonium acetate leachate. The calcium was estimated as oxalate by titrating against standard permanganate solution, potassium by cobalti nitrite method and sodium by magnesium-uraynl-acetate reagent.

The leaching or the washing experiments were carried out in two different ways. In one case, 10 gm. of the salt-treated soil-samples were shaken with 50 c.c. distilled water in glass-stoppered bottles for one hour and kept overnight. Next day, the contents were poured over a Buchner funnel fitted with a filter paper and connected to a filter pump through a filtering flask. The first leachate was collected and then fresh 25 c.c. of distilled water was added on the Buchner funnel. The leachate was again collected and this process was repeated several times. In this way fresh additions of water were continued, leachates obtained and the pH estimated till the leachates showed the pH of water. The first leachate was also analysed for its calcium content. The washed soil was analysed for its exchangeable calcium and potassium contents.

In another set of experiments a modified process was adopted. 20 gm. of salt-treated soil samples were shaken for one hour with 100 c.c. of distilled water and the mixture filtered through a Buchner funnel after 24 hours. The soil from the filter-paper was cautiously washed back to the respective bottles with a jet of distilled water and the volume was made upto 100 c.c. The bottles were shaken again and the leachates obtained after 24 hours. The successive leachates were analysed for their pH values and calcium contents. The washed soils were analysed for their exchangeable bases.

RESULTS

TABLE I (a)
Chemical composition of the soils used

Constituents %	Normal soil	Garden soil
Loss on ignition	3·88	3·81
HCl-insolubles	81·64	79·94
Sesquioxides	11·17	9·86
Fe_2O_3	4·12	4·26
CaO	0·99	2·80
MgO	1·16	1·73
K_2O	0·73	0·98
P_2O_5	0·08	0·12
Total carbonates as $CaCO_3$	1·24	3·95
pH	7·6	8·0

The normal soil was also treated with dilute solutions of potassium chloride, calcium chloride, magnesium chloride and potassium chloride + calcium chloride or potassium chloride + magnesium chloride, potassium dihydrogen phosphate and dipotassium hydrogen phosphate and then washed immediately with distilled water for a number of times. The pH values of the salt-extracts as well as of the successive water-extracts were determined.

TABLE I (*b*)
Exchangeable bases present originally
(m.e./100)

	Ca	Mg	K	Na
Normal soil ..	22·7	3·6	2·4	0·4
Garden soil ..	44·6	3·6	2·0	1·0

In Table II (*a*) the various treatments of the soils are recorded. They were mixed with salts and exposed to light in enamelled dishes. A, B, C, etc., stand for treatments of Normal soil-somples and A_1, B_1, C_1, etc., for Garden soil-samples.

TABLE II (*a*)

Dish No.	CONTENTS
A or A_1	100 g. of normal or garden soil +1·49 g. KCl
B or B_1	,, ,, ,, +1·17 g. NaCl
C or C_1	,, ,, ,, +3·48 g. K_2SO_4
D or D_1	,, ,, ,, +2·84 g. Na_2SO_4
E or E_1	,, ,, ,, +1·06 g. Na_2CO_3
F or F_1	,, ,, ,, +0·84 g. $NaHCO_3$
G	,, ,, soil+0·75 g. KCl +1·42 g. Na_2SO_4
H	,, ,, ,, 0·585 g. NaCl+1·74 g. K_2SO_4
I	,, ,, ,, 0·75 g. KCl+0·05 g. Na_2CO_3
J	,, ,, ,, 0·75 g. KCl+1·00 g. $CaCO_3$
K	,, ,, ,, 0·75 g. KCl+0·084 g. $NaHCO_3$
L	,, ,, ,, 0·75 g. KCl+0·085 g. NaCl
M	,, ,, ,, 0·75 g. KCl+0·585 g. NaCl+1·11 g. $CaCl_2$
N	,, ,, ,, 0·106 g. Na_2CO_3+1·00 g. $CaCO_3$

Note.—The Garden soil was not exposed with the mixtures of salts as this soil was rich i calcium carbonate.

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Table II (*b*)

Chemical analysis of Faizabad soils

Constituents %	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
					Soils					
Loss on ignition	2·82	2·32	2·16	2·69	2·86	1·94	2·11	2·27	1·90	2·40
HCl–insoluble	84·61	84·10	86·38	82·80	83·61	86·26	88·00	87·10	87·73	87·10
Sesquioxides	8·57	9·53	7·90	10·11	9·38	7·75	7·00	6·80	7·35	7·40
CaO	0·61	0·52	0·49	0·69	0·58	0·70	0·72	0·64	0·46	0·76
MgO	1·31	1·24	1·26	1·39	1·39	1·23	1·17	1·10	0·99	1·14
K_2O	1·94	2·31	2·70	2·54	2·36	2·99	1·37	1·24	2·18	1·37
pH	9·7	9·4	9·1	9·6	9·0	9·8	9·7	8·1	9·4	9·4

Table II (*c*)

Exchangeable bases of Faizabad soils

(m.e./100)

Exchangeable bases	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
					Soils					
Ca	13·7	11·0	10·9	16·4	10·8	14·8	15·2	14·0	10·4	14·1
Mg	5·4	5·0	5·7	4·5	2·4	3·9	3·2	2·8	2·8	3·1
Na	2·1	2·9	2·8	3·3	2·7	3·4	3·2	1·9	2·8	2·9
K	2·5	3·5	3·8	2·2	2·4	1·8	2·4	1·9	4·0	3·8

Table III

pH values of the exposed sets after one month

(1 : 2·5 soil-water ratio)

	A_1	B_1	C_1	D_1	E_1	F_1
	7·8	7·9	8·3	8·2	9·85	9·35

A	B	C	D	E	F	G	H	I	J	K	L	M	N
7·7	7·9	8·5	8·3	10·0	9·6	8·3	8·1	8·0	7·9	8·6	7·8	7·5	8·7

TABLE IV (a)

Leaching experiments by the first method and pH values of successive leachates

(10 gm. soil + 50 c.c. water followed by 25 c.c. water each time)

GARDEN SOIL

Leachings No.	A_1	B_1	C_1	D_1	E_1	F_1
	Treatments					
1	7·5	7·6	7·7	7·9	9·2	9·1
2	7·8	8·6	8·9	8·0	9·8	9·0
3	*8·4*	*8·8*	*9·4*	*8·8*	*9·9*	*9·5*
4	8·3	8·5	9·1	8·8	9·7	9·3
5	8·3	8·2	9·1	9·0	9·1	9·2
6	8·3	8·0	8·0	9·3	9·0	9·1
7	7·8	7·6	7·9	9·0	8·9	8·3
8	7·7	7·5	7·8	8·2	8·7	8·2
9	7·5	7·2	7·8	8·0	8·7	8·0
10	7·1	7·0	7·8	7·8	8·5	8·0

TABLE IV (b)

NORMAL SOIL

Leachings No.	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	7·3	7·7	7·4	7·5	9·4	9·3	8·0	7·8	8·1	7·9	8·1	7·7	7·6	8·1
2	7·4	7·9	8·4	8·4	9·7	9·2	8·3	8·0	8·3	8·1	9·1	8·1	7·3	7·9
3	*8·3*	*8·3*	*9·4*	*9·0*	*9·4*	*9·2*	*8·2*	*8·7*	*8·2*	*8·5*	*8·9*	*8·3*	*7·2*	*7·7*
4	7·6	8·2	9·3	8·6	9·2	8·3	8·1	8·9	8·1	8·5	8·8	8·1	7·7	7·6
5	7·2	7·8	8·6	8·1	9·4	8·3	8·0	8·3	8·1	8·2	8·8	8·1	7·7	7·7
6	7·1	7·5	8·5	7·8	9·2	8·4	8·1	8·5	8·0	8·0	8·5	8·1	7·3	7·7
7	7·2	7·3	8·2	7·6	8·2	7·8	8·1	8·2	7·9	7·8	8·4	8·1	7·2	7·5
8	7·0	7·0	7·6	7·5	8·3	7·5	8·0	7·8	7·5	7·5	8·4	7·8	7·1	7·4
9	..	..	7·2	7·0	8·1	7·2	7·8	7·5	7·4	7·4	7·8	7·5	7·0	7·0
10	..	..	..	..	7·9	..	..	..	..	..	7·5	..	..	7·0

TABLE V (a)

Calcium given out in the first three leachates of the salt-treated

NORMAL SOIL SAMPLES (1 : 5)

(m.e./100)

No. of leachates	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	5·14	3·46	5·94	4·75	0·3	0·1	7·62	8·91	5·74	6·66	1·00	6·73	17·42	0·80
2	0·84	0·70	0·60	0·30	..	..	0·79	1·38	1·48	1·08	1·00	0·4	1·48	..
3	0·10	..	..	..	..	..	..	0·20	0·30	0·10	..	..	0·20	..

TABLE V (b)

Calcium given out in GARDEN SOIL *leachates* (1 : 5 *ratio*)

(m.e./100)

No. of leachates	A_1	B_1	C_1	D_1	E_1	F_1
1	7·0	2·97	6·43	4·55	Nil	Nil
2	0·4	0·69	0·39	0·24	..	..

TABLE V (c)

Exchangeable bases in the washed soil samples of the NORMAL SOIL *after various treatments*

(m.e./100)

Exchangeable bases	Treated samples													
	A	B	C	D	E	F	G	H	I	J	K	L	M	N
Ca	16·8	19·99	19·0	16·0	14·0	15·0	15·7	15·4	16·1	21·0	21·1	18·2	23·2	21·1
Mg	3·4	3·4	3·5	3·5	3·6	3·6	3·5	3·5	3·6	3·2	3·5	3·4	2·0	3·1
K	2·80	2·0	2·86	2·0	2·0	2·1	2·42	4·85	3·0	1·61	2·41	1·6	1·5	1·5
Na	0·2	0·8	0·1	0·9	2·4	2·0	0·5	0·2	0·9	0·0	0·5	0·2	0·0	0·8
pH of the washed samples	8·0	7·9	8·0	7·9	8·2	8·0	8·0	7·7	8·0	7·6	7·9	7·6	7·6	7·2

TABLE VI (*a*)

pH values of the leachates in leaching experiments by the second method
(20 g. of soil samples + 100 c.c. water)

No. of leachings	B_1	B	D_1	D	E_1	E
1	7·8	7·6	7·8	7·5	10·2	9·9
2	8·8	7·5	9·1	8·4	10·0	9·8
3	8·8	8·5	9·3	8·7	8·50	9·6
4	8·6	8·1	9·2	8·6	8·40	9·6
5	8·3	8·0	8·6	8·0	8·20	9·4
6	8·2	7·6	8·5	7·7	8·20	8·4
7	7·8	7·5	8·3	7·6	7·8	8·3
8	7·8	7·5	7·9	7·6	7·8	8·2
9	..	7·5	7·7	7·5	..	7·9

TABLE VI (*b*)

Calcium given out in the successive leachates
(m.e./100)

No. of leachings	B_1	B	D_1	D	E_1	E
1	2·97	3·46	4·6	4·8	Nil	0·3
2	0·10	Nil	Nil	Nil	Nil	Nil
3	0·3	0·4	Nil	Nil	Nil	Nil
4	0·6	1·1	0·5	1·2	0·5	0·4
5	0·6	1·1	0·5	0·9	0·6	0·4
6	Nil	Nil	Nil	Nil	Nil	0·35
7	0·7	2·0	0·6	1·1	0·8	Nil
8	Nil	Nil	0·6	1·0	Nil	Nil

Table VII

pH values of the successive leachates

(100 c.c. of the solution + 20 g. Normal Soil shaken, filtered and then 50 c.c. water added)

(a) *M/10 KCl and then water :*
7·2 7·7 8·5 8·4 8·3 8·2 7·1 7·0

M/10 KCl used for all leachates:
7·2 7·4 7·4 7·4 7·4 7·4 7·4 7·4

Water added to the above treated soil :
8·7 8·9 9·1 9·1 8·7 8·6 8·0

M/100 KCl added to the above washed soils :
7·0 7·0 7·2 7·2 7·3

Water added to the above treated soils :
8·7 8·7 8·6 8·3 7·8

(b) *M/10 KCl (50 c.c.) + M/20 CaCl₂ (50 c.c.) and then water :*
6·8 7·3 7·8 7·9 8·1 8·3 8·3 8·2 7·8 7·6 7·6

(c) *M/100 KCl and then water :*
7·1 7·1 7·4 7·4 7·7 7·7 7·6 7·7 7·7

(d) *M/10 CaCl₂ and then water :*
6·8 7·3 7·5 7·5 7·5 7·5

(e) *M/10 MgCl₂ and then water :*
7·0 7·6 8·2 7·95 8·0 7·9 8·0 8·0

(f) *M/10 KH₂PO₄ and then water :*
5·2 5·9 7·0 7·1 7·1 7·1 7·1

(g) *M/100 KH₂PO₄ and then water :*
6·1 6·4 7·2 7·3 7·2 7·2 7·2

(h) *M/10 K₂HPO₄ and then water :*
7·9 8·3 9·0 9·1 8·9 8·8 8·8 8·7 8·7 8·4 8·1 7·7

Discussion

The addition of the salts of sodium and potassium, together with calcium in some cases, the normal soil and its alternate wetting and drying for one month in order to imitate natural conditions under which alkali soils are formed, results in a marked increase in the pH values of the soil-pastes at 1 : 2·5 soil-water ratios. The highest pH values are obtained where sodium carbonate and sodium bicarbonate were added either singly or in combination with other salts. The soil samples E, F, N and K show the pH values in a descending order and with other treatments the order of alkalinity is in the order C>D>G>H>I>J>B>A. The pH values of these samples are much higher than that of normal soil without any treatment (the pH of normal soil being 7·6). The increases in pH values clearly indicate that the accumulation of salts and their alternate wetting and drying are responsible for the observed increases. The effect of sodium carbonate and

bicarbonate in raising the pH values is partly due to their presence in the free-state in the samples and partly due to the entry of sodium ions from carbonates and bicarbonates and a subsequent hydrolysis due to wetting and drying. Also, the sodium ions displace the calcium ions which get precipitated as calcium carbonate. The free alkali dissolves the humus of the soil imparting a black colour to the extracts and also deflocculates the soil particles thereby bringing a cementation as observed in the samples E and E_1 and F and F_1. Also, from Table III, sodium bicarbonate is less effective than sodium carbonate as the pH values are a little less but ultimately sodium bicarbonate is converted into sodium carbonate by hydrolysis.

Next to the cases of soils treated with sodium carbonate and sodium bicarbonate, potassium sulphate-treated-sample C shows the highest pH value. Then comes sodium sulphate-treated-sample. The sodium sulphate and potassium chloride-treated-sample G shows an equal pH value. A combination of potassium sulphate and sodium chloride is inferior to the combination of sodium sulphate and potassium chloride in producing alkalinity. In the samples I and J, greater effect of smaller dose of sodium carbonate in increasing the pH is obvious in comparison to a large dose of calcium carbonate which produces a pH equal to that of sodium chloride-treated sample. The lowest increase in pH is with potassium chloride-treated sample. These observations bring out clearly a fact that sulphates of sodium and potassium increase the pH of normal soil more efficiently than their chlorides. Also, potassium sulphate is more powerful than sodium sulphate. When chlorides and sulphates of potassium and sodium are mixed in equimolar quantities, they show a reverse behaviour. The pH with potassium sulphate and sodium chloride is lower than that with sodium sulphate and potassium chloride. A similar interaction is observed in the sample L, where sodium chloride and potassium chloride together give a pH value which is intermediate between the pH values with potassium chloride and sodium chloride singly (samples A and B). This appears to be due to the preferential adsorption of sodium ions over potassium ions from their mixtures.

Wherever calcium salts have been introduced, *i.e.*, in J, M and N samples, the changes in the pH values are less sudden. The introduction of calcium ions keeps the pH value at lower levels.

Repeated leaching of all the salt-treated samples brings out the fact that the pH of the third leachate is highest except in a few cases, the sample D deserving special attention where higher pH values appear in the sixth leachate. The first leachate is bound to contain most of the soluble salts; hence, hydrolysis of adsorbed sodium or potassium is rather impossible. The hydrolysis on drying and wetting the samples may be different from the hydrolysis produced on washing the salt-treated soil with water. In the second leaching practically all the salts are removed, thereby initiating the hydrolysis and as soon as more water is added, in the third extract we obtain the highest pH values. These pH values are highest in the third extracts of potassium sulphate-treated normal and garden soils, the value being 9·4. The pH of the extracts of sodium carbonate and sodium bicarbonate-treated samples is high and remains constant for a number of leachings. This is due to the fact that the adsorbed sodium is also being hydrolysed slowly besides the free alkali present in the system. Such a release of alkalinity in sodium carbonate-treated soils has been observed by Misra.[14] There is practically no calcium in the soluble form in the extracts of E and E_1 or F and F_1. It is only after the salts have been thoroughly removed that the pH in the water extracts of M/10 KCl-treated normal soil begnis to rise, It is clear, therefore, that

potassium ions are taken up by the soil through an exchange process and then they hydrolyse out as soon as water is added. The experiments were carried on in presence of air and consequently carbon dioxide was adsorbed in the system. The following reactions are likely to take place in these processes:

$$\text{K-salts} + \text{Normal soil} = \text{K-soil} + \text{Ca, Na or Mg salts.}$$

$$\text{K-soil} + H_2O = KOH + \text{H-soil.}$$

$$KOH + CO_2 = KHCO_3.$$

$$2\,KOH + CO_2 + H_2O = K_2CO_3 + 2H_2O.$$
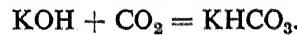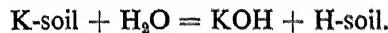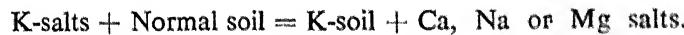
It has been observed that even M/100 potassium chloride solution suppresses the hydrolysis of the adsorbed potassium ions. The presence of calcium ions (as calcium chloride) in the potassium chloride solution greatly decreases the entry of potassium ions into the exchange-complex and, hence, the pH of the extracts does not exceed 8·3 (*vide* Table VII *b*). The amount of calcium present in the various leachates (*vide* Table VI *b*) is in accordance with the above findings. Where the pH is highest, the calcium going into solution is very small.

The addition of dipotassium hydrogen phosphate solution appears to influence the pH values of the leachates next to potassium sulphate treatment in the experiments (*vide* Table VII *h*). This is due to the fact that dipotassium hydrogen phosphate solution is itself alkaline and hence acts like sodium bicarbonate and sodium carbonate. Also, phosphate ions precipitate out the soluble calcium just like the soluble carbonate ions do, hence, there is always greater adsorption of potassium ions from phosphate solution. The sulphate may precipitate the calcium as calcium sulphate but the precipitate is more soluble. The production of alkalinity by the application of tripotassium phosphate and dipotassium hydrogen phosphate to normal soils was observed by Misra.[18]

The calcium going into solution is not only the result of exchange reaction because the sum of the calcium in the water extracts and the exchangeable calcium in the leached soil samples is greater than the total exchangeable calcium of the normal soil. Hence some calcium is coming from the dissolution of calcium carbonate present in the soil by the action of soluble salts added. Greater amount of calcium in the sulphate-treated soils is due to the fact that the sulphates of sodium and potassium dissolve greater amounts of calcium carbonate than their chlorides. Similar observations have been made by Cummins.[15]

Gedroiz[16] reported that potassium ions were more powerful in replacing calcium ions of the exchange complex. Reitemeir[17] observed that potassium ions are five times as efficient as sodium ions. My results obtained with potassium chloride and potassium sulphate clearly demonstrate their superiority over sodium chloride and sodium sulphate respectively in increasing the pH values.

The results reported in the foregoing tables indicate the possibility of potassium soils to be more alkaline than those of sodium soils. It seems probable that some of the alkali soils in India might have been formed by the reaction of potassium salts and their subsequent leaching. Misra[18] has observed that alkali soils from Faizabad (U.P., India) contain a very high potash and the exchangeable potassium constitutes 10–20% of the total exchangeable bases (*vide* Table II *b*). This is certainly a new view-point regarding the formation of alkali soils, which

have been ascribed so far to the influence of sodium bicarbonate and sodium carbonate.

SUMMARY

The addition of salts of sodium or potassium, either singly or in combination and alternate wetting and drying considerably increases the pH of normal soils. Potassium sulphate seems to be more efficient than other salts in increasing the pH. The alkalinity is produced only after the soluble salts have been washed out and the hydrolysis of adsorbed sodium or potassium has started.

Even the solutions of potassium chloride and dipotassium hydrogen phosphate are able to increase the pH of the water leachates. The potassium-soils are produced by potassium salts just as sodium soils by the addition of sodium salts under natural conditions and thus there is a possibility of the existence of potassium soils along with the sodium soils.

REFERENCES

1. MONDESIR, P. De, 1888, " Sur le role du pouvoir des terres dans la formation des carbonates des soude Naturals, " *Compt. rend.,* **160,** 459–62.

2. GEDROIZ, K. K., 1912, " Colloidal chemistry as related to Soil Science, I. Colloidal substances in the soil solution. Formation of sodium carbonate in the soil. Alkali soils and saline soils, " *Zhur. Opit. Agron. (J. Exptl. Agr. Russian),* **13,** 363–412.

3. SIGMOND, A. A. J. De, 1916, *Math es Termeszettud Ertesits,* **34,** 279.

4. DOMINICIS, A. De, 1918, *Terreni Salsi e Terreni Alkaline, Le Staz. Sper. Agr. Ital.,* **41,** 103–61.

5. PRESCOTT, J. A., 1922, "Base exchange and alkalinity in Egyptian soils," *Cairo Sci. J.,* **10,** 58–64.

6. CUMMINS, A. B. AND KELLEY, W. P., 1923, " The formation of sodium carbonate in soils, " *Calif. Agr. Expt. Sta. Tech. Paper* 3, 1–35.

7. BURGESS, P. S., 1928, " Alkali soil studies and methods of reclamation, " *Ariz. Agric. Expt. Sta. Bul.,* **123,** 157–81.

8. IVANOVA, E. N., 1930, " Contribution to the study of Solo di Process in the soils of Western Siberia," *Trudy Pochven. Inst. Imeni Dokuchœva,* Nos. 3–4, 207–84.

9. PRASOLOV, L. I. AND ANTIPOV-KARATAEV, I. N., 1939, " The Chestnut Brown soils," *Soils of the U.S.S.R.,* No. 2, Moscow-Leningrad.

10. PANKOV, A. M. AND SHAVRYGIN, P. I., 1934, " On the composition of the absorption complex of the soils in Primanych area," *Trudy Pochv. Inst. Acad. Nauk. (U.S.S.R.),* **9,** 205–35.

11. ARANY, A. AND KELLEY, W. P., 1928, " Chemical effect of gypsum, sulphur, iron sulphate and alum on alkali soil, " *Hilgardia,* **3,** 393–420.

12. REMEZOV, N. P., 1938, " Exchange capacity and composition of exchangeable cations in the chief soil types, " *Pedology,* (U.S.S.R.), No. 5, 639–94.

13. KELLEY, W. P., 1951, *Alkali soils—Their Formation, Properties and Reclamation,* Reinhold Publ. Corp., 330, West Forty-second Street, New York, U.S.A., pp. 56.

14. MISRA, S. G., 1955, " Repeated adsorption of Na_2CO_3 from solution by a normal soil and its subsequent leachings with distilled water," *Proc. Nat. Acad. Sci.,* **24**, 670–77.

15. CUMMINS, A. B., 1926, " The solubility relationships of calcium carbonate with special reference to the formation of Na_2CO_3 in soils, " *Ph.D. Thesis,* University of California.

16. GEDROIZ, K. K., 1918–19, " Contribution to our knowledge of the absorbing capacity of the soil, " *Zhur. Opit. Agron.,* **19**, 269–322; **20**, 31–58.

17. REITEMEIR, R. F., 1946, " Effect of moisture content on the dissolved and exchangeable ion of the soils of arid regions," *Soil Sci.,* **61**, 195–214.

18. MISRA, S. G., 1954, " Formation of acid and alkali soils, " *D.Phil. Thesis,* Allahabad University.

COMPOSITION OF COBALT THIOCYANATE COMPLEX IN ACETONE

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THE formation of tri- and tetra-thiocyanate complex of cobalt in acetone solutions was noticed by Katzin and Gebert[1] and in relatively concentrated solutions an unstable complex involving single thiocyanate was also detected. Babko and Drako[2] reported the formation of two complexes $Co(CNS)^+$ and $Co(CNS)_4^{--}$ and failed to detect a trithiocyanate complex. In view of the conflicting evidence about the composition of the cobalt thiocyanate complex in acetone, a comprehensive study of this reaction was undertaken and in the present communication the results of the colorimetric study are reported.

All optical density and transmittance measurements were made with a Lumetron photo-electric colorimeter (Photovolt Corporation, New York City, model S–48 A). Cobalt perchlorate was prepared from the carbonate and perchloric acid and recrystallised twice. The solid so obtained was cobalt perchlorate hexahydrate. Lithium thiocyanate was prepared by metathesis of potassium thiocyanate and lithium nitrate in acetone solution. The precipitated potassium nitrate was removed by centrifugation, and the thiocyanate concentration was determined by titration against a silver nitrate standard. All reagents used were of Analar quality.

The monovariation method was used for investigating the nature of the complex formed. In this method the cobalt concentration is kept constant in a given volume of the solution and the lithium thiocyanate concentration was changed upto eight moles of thiocyanate for one mole of cobalt. On applying this technique to mixtures of cobalt perchlorate and lithium thiocyanate in acetone, over a concentration range of $0 \cdot 001 – 0 \cdot 05$ M Co^{++}, the nature of the complex formed was found by plotting the optical density in different filters against the $Co^{++}: CNS^-$ molar ratio.
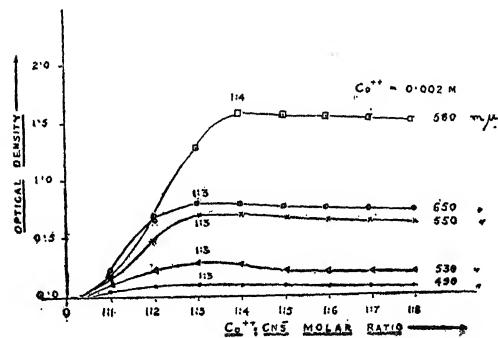

FIG. 1
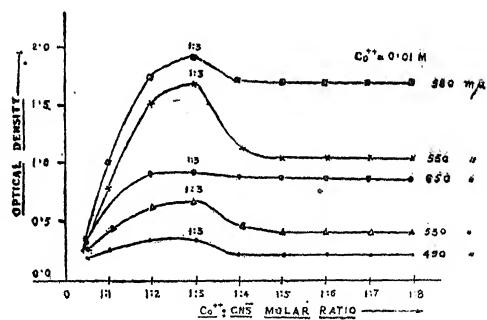
FIG. 2

In dilute solutions of cobalt the formation of trithiocyanate and tetrathiocyanate complexes of cobalt was observed, as seen in Fig. 1. In relatively concentrated solutions of cobalt the trithiocyanate complex only was throughout detected and the tetrathiocyanate complex does not seem to occur in concentrated solutions of cobalt, as can be observed from Fig. 2. Thus our observations do not support the results of Babko and Drako (*loc. cit.*) who did not detect the trithiocyanate complex; whereas the formation of this complex was observed by us throughout the concentration range of 0·001–0·05 M of cobalt and in all the filters used.

REFERENCES

1. Katzin, L. I. and Gebert, E., 1950, *J.A.C.S.*, **72**, 5659.

2. Babko, A. K. and Drako, O. F., 1949, *J. Gen. Chem. U.S.S.R.*, **19**, 1809.

PRELIMINARY STUDIES ON ALIGARH SOIL

Chemical Analysis of Different Soil Samples from the University Campus

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The various physico-chemical properties of Aligarh soil seem to be very much influenced by the defective drainage of the water used for its irrigation. It appears that due to this defect the salts from the waters are continuously being added to soil resulting in an increase in the salinity. Considering this factor in view it was thought worthwhile to carry out an analysis of various soil samples from different areas in the University campus. The results on the amount of various cations like Ca, Al, Fe, Mg, K, Na and ammonium and anions, *viz.*, silicate, CO_3, phosphate and nitrates and on the total water-soluble salts, water-holding capacity, carbon, nitrogen, loss on ignition and pH are communicated in this paper.

EXPERIMENTAL

Soil samples were collected from different places around the University, by digging a hole (diameter 6″) upto 6″, in the ground and obtaining the soil from five such holes dug in the same field. The soil thus obtained was powdered, mixed thoroughly, passed through a 2 mm. sieve and a part of it was dried in air for about a week. Such soil samples were analysed for different constituents.

(1) *Analysis for cations and anions.*—Ca, Na and K were estimated by the Flame photometer (Lange). A weighed amount of the soil was taken and dissolved in concentrated HCl. The solution was analysed for Ca, Na and K by Flame photometer and iron by colorimeter (Lange) using ammonium sulphocyanide as colouring agent. Aluminium was estimated as Al_2O_3 by precipitating $Al(OH)_3$ and $Fe(OH)_3$ and substracting the amount of Fe_2O_3 from the weight of Al_2O_3 and Fe_2O_3. The MCl insoluble portion was weighed and reported as SiO_2. Mg was estimated as pyrophosphate and phosphorus by the ammonium molybdate method. Carbonate was determined by passing the CO_2 evolved (on treatment with H_2SO_4) through standard baryta solution. The procedure adopted for the estimation of nitrate was that of leaching a weighed amount of soil (10 gm.) on a Buchner funnel by dilute H_2SO_4 solution. The leachate was boiled with MgO until it was reduced to half its original volume. The solution was cooled and zinc dust (5 gm.), caustic soda solution (30% 70 c.c.) and powdered iron (5 gm.) added in the above order. The solution was then distilled in a nitrogen distillation apparatus for about one hour and the evolved ammonia was collected in a standard HCl solution. The solution was then titrated with NaOH solution of known concentration and as usual the amount of nitrate was calculated. For ammonia the soil was shaken with N NaCl solution and filtered. The soil was washed with N NaCl and the filtrate was boiled with magnesia and the evolved NH_3 was absorbed in standard HCl and then titrated.

The result for SiO_2, CO_3, CaO, Al_2O_3, Fe_2O_3, MgO, K_2O, Na_2O, P_2O_5, NO_3 and NH_3 are given in Table I.

TABLE I

Soil Sample No.	SiO_2	CO_3	CaO	Al_2O_3	Fe_2O_3	MgO	K_2O	Na_2O	P_2O_5	NO_3	NH_3
1 From a lawn, isolated from adjacent areas	86·7	0·51	1·5	1·63	3·58	0·108	0·082	0·083	0·128	0·0049	0·0018
2 S.-W. of Aligarh, cultivation of gram	88·33	0·77	0·48	2·73	2·7	0·121	0·11	0·351	0·134	0·0061	0·0031
3 N.-W. of Aligarh, Marshy	86·79	0·72	2·72	2·199	3·0	0·213	0·16	0·24	0·0671	0·0018	0·0012
4 W. of Aligarh	83·276	0·802	0·522	2·70	4·6	0·135	9·408	0·428	0·0335	0·0025	0·0014
5 Near a distributary of Canal	90·396	0·832	0·495	1·037	2·73	0·992	0·15	0·395	0·096	0·001	..
6 Soil compact, very hard	78·84	0·61	0·197	2·64	4·68	0·091	0·249	1·90	0·0832	0·0015	0·0011
7 Clay in abundance, dry	83·31	0·306	1·32	0·73	4·26	0·216	0·225	1·16	0·069	0·0022	..
8 From Garden of Fort	88·77	0·41	0·52	2·39	5·64	0·134	0·285	0·32	0·0238	0·0089	0·0062
9 From Bhamola	90·55	0·512	0·40	1·37	2·2	0·210	0·14	0·284	0·0215	0·0056	0·0021
10 From behind V. M. Hall	78·83	0·77	0·88	6·15	10·92	0·248	0·218	0·448	0·071	0·001	0·0026
11 Chaoultala Village	89·691	0·821	0·02	0·27	4·32	0·101	0·15	0·348	0·065	0·0025	..
12 West to the Fort, slopy	85·25	0·853	1·3	..	6·5	0·66	0·856	0·83	0·182	0·0038	0·0026
13 From a lawn	84·25	0·234	0·2	2·97	5·64	0·123	0·46	0·40	0·153	0·0058	0·0031
14 From Agriculture Farm	86·6	1·58	1·21	3·213	2·57	0·336	0·178	0·788	0·965	0·0046	0·0068
15 Behind Minto-Circle	76·6	1·57	0·65	6·64	2·36	0·063	0·82	4·10	0·098	0·0028	0·0013
16 V. M. Hall Lawn	79·62	0·972	0·80	5·029	2·16	0·83	0·63	2·52	0·048	0·0039	0·0036
17 From a cane-sugar field, fertilizer used	72·89	1·957	0·889	5·21	3·12	1·82	0·738	0·283	0·2325	0·0031	0·0013

(2) Total nitrogen was estimated by Kjeldahl's method using salicylic acid reduction method. The total carbon was estimated by the method of Robinson, Mc Leau and Williams (4th Edn. Knowles & Watkins, p. 46). For water-soluble salts, weighed amount of soil was shaken with distilled water and filtered. The filtrate was evaporated to dryness and the solid weighed as total water-soluble salts. For loss on ignition a blank experiment was performed by igniting a weighed amount of the soil.

(3) *The water-holding capacity.*—This was estimated by taking 200 gm. of soil sample on a filter-paper wetting it with water and then taking the weight after 5 hours. For pH measurements Lange's Acidometer was used. The results for total carbon, nitrogen, water-soluble salts, loss on ignition, water-holding capacity and pH are given in Table II.

TABLE II

Sample No.	Total Carbon	Total Nitrogen	Water-soluble salts	Loss on ignition	Water holding capacity	pH
1	0·45	0·0808	0·78	4·71	23·0	6·8
2	0·293	0·0721	0·95	3·57	17·2	7·0
3	0·186	0·0321	0·931	2·6	17·3	6·9
4	0·603	0·076	0·13	5·85	33·2	7·8
5	0·404	0·012	2·73	3·47	22·2	9·3
6	0·586	0·051	0·98	6·54	26·6	7·2
7	0·422	0·063	0·85	3·52	29·0	7·8
8	0·08	0·093	0·35	1·44	39·4	7·6
9	0·095	0·0825	0·39	1·68	45·8	7·3
10	0·34	0·0123	0·687	0·944	64·0	7·4
11	0·15	0·0232	0·0021	1·112	49·6	7·1
12	0·34	0·0543	0·42	2·276	49·2	7·05
13	0·468	0·0768	0·251	4·41	22·3	7·02
14	0·312	0·093	0·231	4·52	28·3	7·22
15	0·326	0·0362	0·624	4·72	53·3	8·47
16	0·263	0·046	0·315	4·86	26·6	7·62
17	0·3121	0·0833	0·187	6·30	23·6	7·3

DISCUSSION

The analysis of various samples tabulated above shows that our soil is deficient in nitrogen, phosphorus and carbon. This may be due to the salinity

of the soil which does not allow perennial grasses to grow in abundance. Hence organic matter and nitrogen which are added by the decay of grasses are not provided to the soil in adequate quantities. The increased amount of water-soluble salts confirms the postulate that the salts of irrigation water are being accumulated in our soil and are not being leached away as the natural drainage is defective and no appropriate steps have been taken to make it efficient artificially.

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